Mercurial Signatures for Variable-Length Messages

Abstract: Mercurial signatures are a useful building block for privacy-preserving schemes, such as anonymous credentials, delegatable anonymous credentials, and related applications. They allow a signature $\sigma$ on a message $m$ under a public key $pk$ to be transformed into a signature $\sigma'$ on an equivalent message $m'$ under an equivalent public key $pk'$ for an appropriate notion of equivalence. For example, $pk$ and $pk'$ may be unlinkable pseudonyms of the same user, and $m$ and $m'$ may be ununlinkable pseudonyms of a user to whom some capability is delegated. The only previously known construction of mercurial signatures suffers a severe limitation: in order to sign messages of length $\ell$, the signer’s public key must also be of length $\ell$. In this paper, we eliminate this restriction and provide an interactive signing protocol that admits messages of any length. We prove our scheme existentially unforgeable under chosen open message attacks (EUF-CoMA) under a variant of the asymmetric bilinear decisional Diffie-Hellman assumption (ABDDH).

Keywords: Signature schemes, anonymous credentials.

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1 Introduction

Suppose Alice is known by a public key $pk_{Alice}$, and Bob is known by a public key $pk_{Bob}$. Suppose also that Alice has a certificate on her public key and relevant attributes from some certification authority (CA). Attributes may include the expiration date of the certificate or information about resources to which a user has been granted access. Alice’s certificate consists of her public key $pk_{Alice}$ and attributes $attr_{Alice}$ and a signature on them from the CA: $\sigma_{CA\rightarrow Alice}$, as well as his own public key $pk_{Bob}$ and attributes $attr_{Bob}$ and certificate from Alice, $\sigma_{Alice\rightarrow Bob}$.

A conventional signature scheme allows Alice to certify Bob as above. However, a mercurial signature allows the signer, Alice, to sign a message, such as Bob’s public key and attributes, with two important blinding features that make it attractive in privacy-preserving applications. The first feature is message-blinding: the original message $m$ and its corresponding signature $\sigma$ can be transformed into an equivalent message $m'$ and corresponding signature $\sigma'$. The second feature is public key-blinding, which allows the original public key and corresponding signature to be transformed as well.

Let us see how these two privacy-preserving features may be used in the scenario above. Mercurial signatures allow Bob to transform the public keys on his certification chain and derive valid signatures for these transformed values. Specifically, he can transform $pk_{Alice}$ into an equivalent $pk'_{Alice}$, where Alice’s secret key will also correspond to this new public key. Bob can then adapt $\sigma_{CA\rightarrow Alice}$ into $\sigma'_{CA\rightarrow Alice}$, which is the CA’s signature on the transformed public key $pk'_{Alice}$ and attributes $attr_{Alice}$. This can be done using the message-blinding feature of mercurial signatures. Using the public key-blinding feature, Bob can also adapt $\sigma_{Alice\rightarrow Bob}$ into $\sigma'_{Alice\rightarrow Bob}$, which is a valid signature on $pk'_{Alice}$ and attributes $attr_{Bob}$. He can then repeat the process to transform his own public key $pk_{Bob}$ into an equivalent but unlinkable $pk'_{Bob}$ and derive the corresponding signature $\tilde{\sigma}_{Alice\rightarrow Bob}$. It is easy to see that this can be extended to longer certification chains. These blinding features are desirable because certificate holders do not have to disclose all of the information on their certification chains every time they use them. In particular, the public keys on certification chains are blinded, concealing the identities of the users operating under them.

Mercurial signatures were introduced in a recent paper by Crites and Lysyanskaya [15]. The construction consists of messages and public keys that are vectors of group elements of a certain fixed length. Specifically, messages and public keys are of the form $M = (M_1, \ldots, M_\ell)$ and $pk = (X_1, \ldots, X_\ell)$ for a fixed length $\ell$, where $M$ and $pk$ are defined over bilinear groups $G_1$ and $G_2$, respectively. Mercurial signatures allow a message $M$ to be transformed into an equivalent message...
\(M' = (M_1^\mu, \ldots, M_{\ell}^\mu)\) using a scalar \(\mu\), and public keys may be transformed similarly.

We present a construction of mercurial signatures that inherits this structure but allows for messages of unbounded length. A message space that consists of vectors of any length is very convenient because, in particular, it allows for signatures on public keys and any number of attributes. Consider anonymous credentials, wherein users receive credentials directly from the CA. (Such directly issued credentials are referred to as level-1 credentials.) Suppose Alice’s public key is \(\text{pk}_{\text{Alice}} = (\hat{X}_1, \ldots, \hat{X}_\ell)\) and her attributes are some values \((a_1, \ldots, a_k)\) that represent access to a particular set of buildings at particular times. If Alice intends to reveal her attributes every time she uses her certificate, she may encode them as \((\hat{X}_1^{a_1}, \ldots, \hat{X}_1^{a_k})\) and simply append them to the vector representing her public key. (Of course, a limitation of encoding attributes this way is that they are exposed. In this paper, we do not address limited disclosure of attributes.) Her certificate is then the CA’s signature on this combined vector \(M = (\hat{X}_1, \ldots, \hat{X}_\ell, \hat{X}_1^{a_1}, \ldots, \hat{X}_1^{a_k})\) of length \(\ell + k\). If the message is transformed into an equivalent message \(M' = (\hat{X}_1^{\mu}, \ldots, \hat{X}_\ell^{\mu}, \hat{X}_1^{\mu a_1}, \ldots, \hat{X}_1^{\mu a_k})\), the attributes remain the same relative to the base \(\hat{X}_1^\mu\), so Alice’s certificate still authorizes access to the same buildings at the same times. A message space that consists of vectors of any length is also desirable because the CA does not need to know how many attributes a user has ahead of time.

Now consider delegatable anonymous credentials, wherein a user receives a level-L credential from a level-L-1 user. In particular, suppose Alice issues a level-2 credential to Bob that grants him access to the same buildings or a subset of the buildings to which she has access, potentially limiting the hours during which Bob is authorized. Under the mercurial signature scheme of [15], if Alcice’s public key is of length \(\ell\) and her attributes are of length \(k\), the CA’s public key must be of length \(\ell + k\). This, in turn, severely limits the kinds of key–attribute pairs that Alice can sign with a public key of length \(\ell\) and Bob can sign with a public key of length \(\ell = |\text{attr}_{Bob}|\) (and so on down the chain). Furthermore, while the construction of [15] permits this kind of delegation, the proofs of security do not. Delegatable anonymous credentials in [15] are proven secure only when all public keys and messages are of the same fixed length \(\ell\). The mercurial signature scheme presented in this work allows messages to include any number of elements.

1.1 Related Work and Applications

Our motivating application is anonymous credentials [5–8, 14, 24, 25]. In an anonymous credential system, users can obtain credentials anonymously as well as prove possession of credentials without revealing any other information (via zero-knowledge proofs). Anonymous credentials are well studied and have been incorporated into industry standards (such as the TCG standard [4]) and government policy (such as the NSTIC document released by the Obama administration).

Mercurial signatures are a natural building block for anonymous credentials. In order to anonymously obtain a credential, Alice requests a signature from the CA on one of her many equivalent public keys. In order to anonymously use her credential, Alice blinds her public key and the CA’s signature and gives a zero-knowledge proof of knowledge (ZKPoK) of the secret key corresponding to her public key. Crucially, it is difficult to distinguish whether or not a pair of public keys (and thus identities) are equivalent.

Mercurial signatures are used as a building block for even more interesting applications, such as delegatable anonymous credentials [15]. In this setting, a participant may use her credential anonymously as well as anonymously delegate it to others, all while remaining oblivious to the true identities of the users on her credential chain. All prior constructions of delegatable anonymous credentials relied on costly non-interactive zero-knowledge (NIZK) proofs [2, 12, 13], such as Groth-Sahai proofs [23], which made them too inefficient for practical use. (Some required hundreds of group elements to represent a chain of length two.) Mercurial signatures allow for modular constructions of delegatable anonymous credentials that do not require NIZKs and are substantially more efficient: in the construction of [15], only five group elements are needed to represent each link in a credential chain.

A user may in fact be in possession of several types of credentials: a credential issued by her employer, one issued by the government, and another issued by a service provider, for example. Multi-authority delegatable anonymous credentials allow users to anonymously obtain, demonstrate possession of, and delegate credentials under different certification authorities, all with the same underlying identity. For example, suppose Alice has a level-1 credential from her employer and a level-2 credential from the government. Under the mer-
curial signature scheme of [15], Alice could not possess a single underlying secret key. To see why, suppose a credential has just one single attribute. Following [15], to give Alice a level-1 credential, her employer signs a representative of the equivalence class of her public keys. If her secret key is a vector of length \( \ell \), then her public key is also a vector of length \( \ell \). To issue a level-2 credential, Alice signs a representative of the equivalence class of Bob’s public keys; however, using a length-\( \ell \)-1 key, she may only sign length-\( \ell \)-1 vectors, so Bob’s secret key must be shorter than Alice’s. By the same logic, any user who has a level-2 credential must have a secret key of shorter length than that of a user with a level-1 credential (under the same CA). In particular, if Alice’s level-2 government credential chain is government \( \rightarrow \) Carol \( \rightarrow \) Alice for some user Carol with a secret key also of length \( \ell \), then Alice’s secret key would need to be of length \( \ell - 1 \). This is a contradiction since Alice’s secret key is of length \( \ell \).

Mercurial signatures for variable-length messages allow users to have the same underlying secret key under different CAs as well as any number of attributes (although note that delegators at the same level must have the same number of attributes or else their signatures are trivially distinguishable). This is a first step towards achieving efficient multi-authority delegatable anonymous credentials.

Mercurial signatures were inspired by Fuchsbauer, Hanser and Slamanig’s work on structure-preserving signatures on equivalence classes (SPS-EQ) [22], which introduces the idea of transforming a signature \( \sigma \) on a fixed-length message \( m \) into a signature \( \sigma' \) on an equivalent but unlinkable message \( m' \). Mercurial signatures [15] additionally allow the fixed-length public key \( pk \) to be transformed into an equivalent public key \( pk' \), where \( pk \) and \( pk' \) are unlinkable even when given signatures under both keys. A related concept, signatures with flexible public key [1], allows blinding of the public key, but not the message.

### 1.2 Our Contribution

The only previously known construction of mercurial signatures [15] was restricted to messages of fixed length, which limits its use in applications. Thus, our goal was to construct mercurial signatures that allow messages of any length to be signed under public keys of a small, fixed length. This is desirable because public keys are pseudonyms, which users may wish to be shorter than the messages they are signing.

While the prior construction of mercurial signatures [15] achieved the standard notion of unforgeability, namely existential unforgeability under chosen message attacks (EUF-CMA), the construction presented here is unforgeable in a more limited sense. Instead of the adversary having access to the usual signing oracle that simply responds with its signature \( \sigma \) on input a message \( m \), the adversary obtains signatures via a signing protocol in which it is required to prove knowledge of the discrete logarithm of each message vector component. This proof of knowledge is needed for proving unforgeability. (The reduction must use these discrete logarithms.) This variant of unforgeability was defined by Fuchsbauer and Gay [20] as existential unforgeability under chosen open message attacks (EUF-CoMA).

The construction of mercurial signatures presented here also differs from that of [15] as far as message-blinding is concerned. Recall that Bob needs to blind a message \( m \) signed by a potentially malicious Alice by transforming it into a new message \( m' \) and adapting her signature \( \sigma \) into \( \sigma' \) accordingly. A property of mercurial signatures called origin-hiding guarantees that the resulting signature \( \sigma' \) is distributed identically to what Bob would have received had \( m' \) been signed anew. Our construction guarantees origin-hiding if the signer follows the signing algorithm, but a malicious signer could issue improperly formed signatures that would allow it to tell whether \( \sigma' \) was adapted from \( \sigma \) or was freshly issued. To mitigate this, the signer convinces the recipient that the signature was formed properly via an efficient zero-knowledge proof as part of the signing protocol.

Though our construction satisfies a weaker notion of unforgeability and origin-hiding, for the purpose of anonymous credentials, our results constitute a success. This is because the protocol for issuing anonymous credentials typically requires that the recipient prove knowledge of her secret key anyway, so relaxing unforgeability to EUF-CoMA comes for free. Relaxing origin-hiding so it holds only when signatures were issued properly adds an additional step to the signing protocol; however, it can be executed efficiently and is therefore also a reasonable relaxation.

Our construction of variable-length mercurial signatures uses the fixed-length mercurial signature scheme of [15] as a building block and is proven secure (under the variants of unforgeability and origin-hiding above) assuming (1) the security of the underlying mercurial signature scheme and (2) the ABDDH\(^*\) assumption, which was introduced in [21] and is reminiscent of the decisional Diffie-Hellman assumption for Type III bilinear pairings.
Towards constructing variable-length mercurial signatures. A naive approach to extending mercurial signatures to allow messages of any length would be to hash the messages down to the correct fixed length and use the fixed-length mercurial signature scheme of [15]. In general, this does not work because we do not readily have a hash function \( H \) such that \( H(m) \) and \( H(m') \) are equivalent when \( m \) and \( m' \) are equivalent.

In order to maintain the equivalence relation among messages, we instead break a message \( m = (\hat{g}, u_1, \ldots, u_n) \), where \( \hat{g} \) is a base group element and \( u_i = \hat{g}^{m_i} \) for some \( m_i \in \mathbb{Z}_p^* \), into its \( n \) constituent group elements \( u_i \). Each \( u_i \), together with powers of a base \( \hat{g} \) indicating the index \( i \), is signed using the fixed-length mercurial signature scheme. However, an adversary may be able to mix and match elements of the \( n \) new messages being signed under the fixed-length scheme. To mitigate this, an additional group element is included in each of the \( n \) messages to link them together and to the original message \( m \) in an unforgeable way. We call this additional element the "glue" element. Specifically, we represent the message \( m \) to be signed as a sequence of \( n \) messages \( M_1 = (\hat{g}, \hat{g}^a, \hat{g}^b, u_1) \), \( M_2 = (\hat{g}, \hat{g}^2, \hat{g}^b, u_2) \), \ldots, \( M_n = (\hat{g}, \hat{g}^n, \hat{g}^b, u_n) \). This allows the message \( m \) to be transformed into an equivalent message \( m' = m^{\mu} = (\hat{g}^{\mu}, u_1^{\mu}, \ldots, u_n^{\mu}) \), for any \( \mu \in \mathbb{Z}_p^* \), by simply changing each \( M_i \) to \( M'_i = M_i^{\mu} = (\hat{g}^{\mu}, (\hat{g}^a)^{\mu}, (\hat{g}^b)^{\mu}, u_i^{\mu}) \) and invoking the underlying algorithm of the fixed-length scheme that updates the signature. The problem with this approach, however, is that different signatures receive different glue values, so origin-hiding does not hold in a statistical sense. In order to satisfy the origin-hiding property, the glue element \( \hat{g}^a \) must be computed (relative to \( \hat{g} \)) as a function of the entire equivalence class to which the message belongs. That way, no matter which message in the class is signed, the glue element's discrete logarithm base \( \hat{g} \) is the same. Our main technical insight is how to compute the glue element such that it is a function of the entire equivalence class that a message represents, and not just the message itself.

2 Preliminaries

A function \( \nu : \mathbb{N} \to \mathbb{R} \) is called negligible if for all \( c > 0 \), there exists a \( k_0 \) such that \( \nu(k) < \frac{1}{k^c} \) for all \( k > k_0 \). Let \( y = A(x) \) denote running a probabilistic algorithm \( A \) on input \( x \) and assigning the output to \( y \).

Definition 1 (Bilinear pairing). Let \( \mathcal{G}_1, \mathcal{G}_2 \), and \( \mathcal{G}_T \) be multiplicative groups of prime order \( p \), and let \( P \) and \( \hat{P} \) be generators of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), respectively. A bilinear pairing is a map \( e : \mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}_T \) that satisfies (1) bilinearity: \( e(P^a, \hat{P}^b) = e(P, \hat{P})^{ab} = e(P^a, \hat{P}) \forall a, b \in \mathbb{Z}_p \); (2) non-degeneracy: \( e(P, \hat{P}) \neq 1_{\mathcal{G}_T} \) (i.e., \( e(P, \hat{P}) \) generates \( \mathcal{G}_T \)); and (3) computability: there exists an efficient algorithm to compute \( e \).

Bilinear pairings can be classified into three types. We consider Type III (asymmetric) pairings, where \( \mathcal{G}_1 \neq \mathcal{G}_2 \) and there is no efficiently computable homomorphism between them.

Definition 2 (Bilinear group generator). A bilinear group generator \( \text{BGGen} \) is a (possibly probabilistic) polynomial-time algorithm that takes as input a security parameter \( k \) and outputs a bilinear group description \( \mathcal{B}G = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_T, P, \hat{P}, e) \) with a Type III pairing.

Definition 3 (Discrete logarithm assumption (DL)). Let \( \mathcal{B}G \) be a bilinear group generator that outputs \( \mathcal{B}G = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_T, P_1, P_2, e) \). For \( i \in \{1, 2\} \), the discrete logarithm assumption holds in \( \mathcal{G}_i \) for \( \text{BGGen} \) if for all probabilistic, polynomial-time (PPT) adversaries \( \mathcal{A} \), there exists a negligible function \( \nu \) such that: \( \Pr[\mathcal{B}G \leftarrow \text{BGGen}(1^k), x \leftarrow \mathbb{Z}_p, x' \leftarrow \mathcal{A}(\mathcal{B}G, P^x) : P_i^x = P_i^{x'}] \leq \nu(k) \).

Definition 4. (Decisional Diffie-Hellman assumption (DDH)). Let \( \mathcal{B}G \) be a bilinear group generator that outputs \( \mathcal{B}G = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_T, P_1, P_2, e) \). For \( i \in \{1, 2\} \), the decisional Diffie-Hellman assumption holds in \( \mathcal{G}_i \) for \( \mathcal{B}G \) if for all probabilistic, polynomial-time (PPT) adversaries \( \mathcal{A} \), there exists a negligible function \( \nu \) such that: \( \Pr[b \leftarrow \{0, 1\}, \mathcal{B}G \leftarrow \text{BGGen}(1^k), s, t, r \leftarrow \mathbb{Z}_p, b^* \leftarrow \mathcal{A}(\mathcal{B}G, P_1^s, P_1^t, P_1^{(1-b)r + b s}) : b^* = b] = \frac{1}{2} \leq \nu(k) \).

3 Definition

We begin with the definition of mercurial signatures. The following definition is mostly a restatement of [15] with a few adaptations to accommodate messages of any length. We denote by \( \mathcal{M}_m \) the message space consisting of all message vectors of length \( n \). The key generation algorithm \( \text{KeyGen} \) no longer takes as input a fixed length parameter, and the signature conversion algorithm \( \text{ConvertSig} \) now takes as input a message converter \( \mu \) to transform \( (m, \sigma) \) into \( (m', \hat{\sigma}) \). The original
construction of mercurial signatures [15] satisfies this revised definition for a fixed-length message space.

**Definition 5** (Mercurial signature). A **mercurial signature scheme** for parameterized equivalence relations $R_m, R_{pk}, R_{sk}$ is a tuple of the following polynomial-time algorithms, which are deterministic algorithms unless stated otherwise:

$PPGen(1^k) \rightarrow PP$: On input the security parameter $1^k$, this probabilistic algorithm outputs the public parameters $PP$. This includes parameters for the parameterized equivalence relations $R_m, R_{pk}, R_{sk}$ so they are all well defined. It also includes parameters for the algorithms sample, and sample, which sample key and message converters, respectively.

$KeyGen(PP) \rightarrow (pk, sk)$: On input the public parameters $PP$, this probabilistic algorithm outputs a key pair $(pk, sk)$. This algorithm also defines a correspondence between public and secret keys: we write $(pk, sk) \in KeyGen(PP)$ if there exists a set of random choices that $KeyGen$ could make that would result in $(pk, sk)$ as the output.

$Sign(sk, m) \rightarrow \sigma$: On input the signing key $sk$ and a message $m \in M$, this probabilistic algorithm outputs a signature $\sigma$.

$Verify(pk, m, \sigma) \rightarrow 0/1$: On input the public key $pk$, a message $m$, and a purported signature $\sigma$, output 0 or 1.

$ConvertSK(sk, \rho) \rightarrow \tilde{sk}$: On input $sk$ and a key converter $\rho \in sample_\rho$, output a new secret key $\tilde{sk} \in [sk]_{R_{sk}}$.

$ConvertPK(pk, \rho) \rightarrow \tilde{pk}$: On input $pk$ and a key converter $\rho \in sample_\rho$, output a new public key $\tilde{pk} \in [pk]_{R_{pk}}$. (Correctness of this operation, defined below, will guarantee that if $pk$ corresponds to $sk$, then $pk$ corresponds to $\tilde{sk} = ConvertSK(sk, \rho)$.)

$ChangeRep(pk, m, \sigma, \mu) \rightarrow (m', \sigma')$: On input $pk$, a message $m$, a signature $\sigma$, and a message converter $\mu \in sample_\mu$, this probabilistic algorithm computes a new message representative $m' \in [m]_{R_m}$ and a new signature $\sigma'$ and outputs $(m', \sigma')$. (Correctness of this will require that whenever $Verify(pk, m, \sigma) = 1$, it will also be the case that $Verify(\tilde{pk}, m', \tilde{\sigma}) = 1$, where $\tilde{pk} = ConvertPK(pk, \rho)$.)

$ConvertSig(pk, m, \sigma, \rho, \mu) \rightarrow (m', \tilde{\sigma})$: On input $pk$, a message $m$, a signature $\sigma$, a key converter $\rho \in sample_\rho, and a message converter \mu \in sample_\mu$, this probabilistic algorithm computes a new message representative $m' \in [m]_{R_m}$ and a new signature $\tilde{\sigma}$ and outputs $(m', \tilde{\sigma})$. (Correctness of this will require that whenever $Verify(pk, m, \sigma) = 1$, it will also be the case that $Verify(\tilde{pk}, m', \tilde{\sigma}) = 1$, where $\tilde{pk} = ConvertPK(pk, \rho)$.)

**Definition 6** (Correctness). A mercurial signature scheme $(PPGen, KeyGen, Sign, Verify, ConvertSK, ConvertPK, ChangeRep, ConvertSig)$ for parameterized equivalence relations $R_m, R_{pk}, R_{sk}$ is correct if it satisfies the following conditions for all $k$, for all $PP \in PGen(1^k)$, and for all $(pk, sk) \in KeyGen(PP)$:

**Verification**: For all $m \in M$, for all $\sigma \in Sign(sk, m)$, $Verify(pk, m, \sigma) = 1$.

**Key conversion**: For all $\rho \in sample_\rho$, $(ConvertPK(pk, \rho), ConvertSK(sk, \rho)) \in KeyGen(PP)$. Moreover, $ConvertSK(sk, \rho) \in [sk]_{R_{sk}}$ and $ConvertPK(pk, \rho) \in [pk]_{R_{pk}}$.

**Change of message representative**: For all $m \in M$, for all $\sigma$ such that $Verify(pk, m, \sigma) = 1$, for all $\mu \in sample_\mu$, for all $(m', \sigma') \in ChangeRep(pk, m, \sigma, \mu)$, $Verify(pk, m', \sigma') = 1$, where $m' \in [m]_{R_m}$.

**Signature conversion**: For all $m \in M$, for all $\sigma$ such that $Verify(pk, m, \sigma) = 1$, for all $\rho \in sample_\rho$, for all $\mu \in sample_\mu$, for all $(m', \tilde{\sigma}) \in ConvertSig(pk, m, \sigma, \rho, \mu)$, $Verify(ConvertPK(pk, \rho), m', \tilde{\sigma}) = 1$, where $m' \in [m]_{R_m}$.

Correct verification, key conversion, and change of message representative are exactly as in [15]. Correct signature conversion means that if a key converter $\rho$ is applied to a public key $pk$ to obtain an equivalent $\tilde{pk}$, and the same $\rho$ together with a message converter $\mu$ is applied to a valid message-signature pair $(m, \sigma)$ to obtain $(m', \tilde{\sigma})$, then the signature $\tilde{\sigma}$ is valid on the equivalent message $m'$ under the public key $pk$.

**Definition 7** (Unforgeability). A mercurial signature scheme $(PPGen, KeyGen, Sign, Verify, ConvertSK, ConvertPK, ChangeRep, ConvertSig)$ for parameterized equivalence relations $R_m, R_{pk}, R_{sk}$ is unforgeable if for all probabilistic, polynomial-time (PPT) algorithms $A$ having access to a signing oracle, there exists a negligible function $\nu$ such that:

$$Pr[PP \leftarrow PGen(1^k); (pk, sk) \leftarrow KeyGen(PP); (Q, pk^*, m^*, \sigma^*) \leftarrow A^{Sign(sk)}(pk); \forall m \in Q, [m^*]_{R_m} \neq [m]_{R_m} \land [pk^*]_{R_{pk}} = [pk]_{R_{pk} \land Verify(pk^*, m^*, \sigma^*) = 1} \leq \nu(k)$$

where $Q$ is the set of discrete logarithms $\bar{m}$ of messages $m$ that $A$ has queried to the signing oracle.

This definition is similar to existential unforgeability under chosen open message attacks (EUF-CoMA) defined by Fuchsbauer and Gay [20]. EUF-CoMA differs from
EUF-CMA in that the adversary must provide the discrete logarithm \( \tilde{m} \) of the message \( m \) to be signed. This has the advantage that the adversary’s success is efficiently verifiable [20]. Our notion of unforgeability is similar to EUF-CoMA, except the adversary’s winning condition is slightly altered. As in the EUF-CoMA game, the adversary is given the public key \( pk \) and is allowed to query the signing oracle that knows the corresponding secret key \( sk \). Eventually, the adversary outputs a public key \( pk^* \), a message \( m^* \), and a purported signature \( \sigma^* \). Unlike the EUF-CoMA game, the adversary has the freedom to output a forgery under a different public key \( pk^* \), as long as \( pk^* \) is in the same equivalence class as \( pk \). This seemingly makes the adversary’s task easier. At the same time, the adversary’s forgery is not valid if the message \( m^* \) is in the same equivalence class as a previously queried message \( m \), making the adversary’s task harder. The definition of unforgeability for mercurial signatures in [15] allows a forgery under an equivalent public key, but does not require the adversary to provide the discrete logarithm of the message to be signed by the oracle.

**Remark.** In Section 4.1, we define an interactive signing protocol in which the recipient of the signature gives a zero-knowledge proof of knowledge (ZKPoK) of the discreteness of the logarithm of the message.

**Definition 8** (Class- and origin-hiding). A mercurial signature scheme \((\text{PPGen}, \text{KeyGen}, \text{Sign}, \text{Verify}, \text{ConvertSK}, \text{ConvertPK}, \text{ChangeRep}, \text{ConvertSig})\) for parameterized equivalence relations \( \mathcal{R}_m, \mathcal{R}_{pk}, \mathcal{R}_{sk} \) is **class-hiding** if it satisfies the following two properties:

**Message class-hiding:** For all polynomial-length parameters \( n(k) \), and for all probabilistic, polynomial-time (PPT) algorithms \( A \), there exists a negligible function \( \nu \) such that:

\[
\Pr[PP \leftarrow \text{PPGen}(1^k); m_1 \leftarrow \mathcal{M}_{n(k)}; m_2^0 \leftarrow \mathcal{M}_{n(k)}; \]
\[
m_2^0 \leftarrow \{m_1 \in \mathcal{R}_m; \ b \leftarrow \{0, 1\}; \ b' \leftarrow A(PP, m_1, m_2^0); \ b = b' \leq \frac{1}{2} + \nu(k) \]
\]

**Public key class-hiding:** For all probabilistic, polynomial-time (PPT) algorithms \( A \), there exists a negligible function \( \nu \) such that:

\[
\Pr[PP \leftarrow \text{PPGen}(1^k); (pk_1, sk_1) \leftarrow \text{KeyGen}(PP); \]
\[
(pk_2, sk_2) \leftarrow \text{KeyGen}(PP); \ \rho \leftarrow \text{sample}_\rho(PP); \]
\[
 pk_1^0 = \text{ConvertPK}(pk_1, \rho); \ sk_1^0 = \text{ConvertSK}(sk_1, \rho); \]
\[
b \leftarrow \{0, 1\}; \ b' \leftarrow A(\text{Sign}(sk_1^0), \text{Sign}(sk_2^0))(pk_1, pk_2^b); \]
\[
b = b' \leq \frac{1}{2} + \nu(k) \]

A mercurial signature is also **origin-hiding** if the following two properties hold:

**Origin-hiding of ChangeRep:** For all \( k \), for all \( PP \in \text{PPGen}(1^k) \), for all \( pk^* \) (in particular, adversarially generated ones), for all \( m, \sigma \), if \( \text{Verify}(pk^*, m, \sigma) = 1 \), if \( \mu \leftarrow \text{sample}_\mu \), then with overwhelming probability \( \text{ChangeRep}(pk^*, m, \sigma, \mu) \) outputs a uniformly random \( m' \in [m]_{\mathcal{R}_m} \) and a uniformly random \( \tilde{\sigma} \in \{\tilde{\sigma} | \text{Verify}(pk^*, m', \tilde{\sigma}) = 1\} \).

**Origin-hiding of ConvertSig:** For all \( k \), for all \( PP \in \text{PPGen}(1^k) \), for all \( pk^* \) (in particular, adversarially generated ones), for all \( m, \sigma \), if \( \text{Verify}(pk^*, m, \sigma) = 1 \), if \( \rho \leftarrow \text{sample}_\rho \) and \( \mu \leftarrow \text{sample}_\mu \), then with overwhelming probability \( \text{ConvertSig}(pk^*, m, \sigma, \rho, \mu) \) outputs a uniformly random \( m' \in [m]_{\mathcal{R}_m} \) and a uniformly random \( \tilde{\sigma} \in \{\tilde{\sigma} | \text{Verify}(\text{ConvertPK}(pk^*, \rho), m', \tilde{\sigma}) = 1\} \). \( \text{ConvertPK}(pk^*, \rho) \) outputs a uniformly random element of \([pk^*]_{\mathcal{R}_{pk}}\).

**Remark.** This definition of origin-hiding is a relaxation of the prior definition [15] in that there is a small probability that the outputs of ChangeRep and ConvertSig are not distributed correctly. It will become clear why in Section 4.1.

### 4 Construction

Let \( G_1, G_2, \) and \( G_T \) be multiplicative groups of prime order \( p \) with a Type III bilinear pairing \( e: G_1 \times G_2 \to G_T \). Similar to the prior mercurial signature scheme [15], the message space for our new mercurial signature scheme consists of vectors of group elements from \( G_1 \), where \( G_1^* = G_1 \setminus \{1_G\} \). Unlike the prior scheme, these can be vectors of any length. The message space is \( \mathcal{M}_n = \{(g, u_1, \ldots, u_n) \in (G_1)^{n+1}\} \), where \( g \) is a generator of \( G_1 \), and for all \( 1 \leq i \leq n, u_i = g^{m_i} \) for some \( m_i \in Z_p^* \). The space of secret keys consists of vectors of elements from \( Z_{p^*}^n \). The space of public keys, similar to the message space, consists of vectors of group elements from \( G_2 \). A scheme with messages over \( G_2^* \) and public keys over \( G_1^* \) can be obtained by simply switching \( G_1^* \) and \( G_2^* \) throughout. Once the prime \( p \), \( G_1 \), and \( G_2 \) are well defined, for a length parameter \( n \in \mathbb{N} \) the equivalence relations are as follows:

\[
\mathcal{R}_m = \{(m, m') \in (G_1^*)^{n+1} \times (G_1^*)^{n+1} | \exists \mu \in Z_p^* \text{ s.t. } m' = m^\mu\}
\]

\[
\mathcal{R}_s = \{(sk_x, sk_x) \in (Z_{p^*}^n)^{10} \times (Z_{p^*}^n)^{10} | \exists \rho \in Z_p^* \text{ s.t. } sk = \rho \cdot sk\}
\]

\[
\mathcal{R}_{pk} = \{(pk_x, pk_x) \in (G_2)^{10} \times (G_2)^{10} | \exists \rho \in Z_p^* \text{ s.t. } pk = pk^\rho\}
\]

Our variable-length mercurial signature scheme, denoted MSX, is an extension of the prior fixed-length scheme, denoted MSF [15], which can be found in Ap-
Appendix A. The subscript X, for extension, is used to denote all keys and algorithms associated with the variable-length scheme MSX.

Let us discuss the security properties of the fixed-length scheme MSf. It satisfies the definition of security in Section 3, but only for the fixed-length message space \( M_5 = (G_1^*)^5 \). If given as input a message \( m \notin M_5 \), the signing algorithm rejects. Correspondingly, correctness only holds for messages of the correct length. MSf satisfies the definition of unforgeability in Section 3 as well as message and public key hiding. As for origin-hiding, ChangeRepf(pk, m, σ, μ) outputs \((m', σ')\), where \( m' = μ^m \in [m]_{R_m} \) for a message converter \( μ \in Z_p^* \) and \( σ' \) is a valid signature on \( m' \) under pk, and ConvertSigf(pk, m, σ, ρ) outputs \( χ \), where \( χ \) is a valid signature on \( m \) under pk = pkρ ∈ [pk]_{R_m} for a key converter ρ ∈ Z_p^*. Both ChangeRepf and ConvertSigf satisfy origin-hiding with probability 1. The following theorem summarizes the security properties of MSf.

**Theorem 1.** [15]. The mercurial signature scheme MSf is correct for fixed-length messages, unforgeable, and satisfies class- and origin-hiding in the generic group model for Type III bilinear groups.

MSX can be constructed from MSf on messages of length \( ℓ = 5 \) as follows. A message \( m \) is written as \( m = (g, u_1, \ldots, u_n) \in (G_1^*)^{n+1} \), where \( g \) is a generator of \( G_1 \) and for all \( 1 \leq i \leq n, u_i = ˜g^{m_i} \) for some \( m_i \in Z_p^* \). For a generator \( ˜g \) of \( G_1 \) and "glue" element \( ˜h \in G_1^* \) (discussed shortly), the message \( m \) can be represented as a set of \( n \) messages that are in the message space of the mercurial signature scheme MSf as follows, where \( ˜u_i = ˜g^{m_i} \) for all \( 1 \leq i \leq n \):

\[
\begin{align*}
M_1 &= (g, ˜g^1, ˜g^n, ˜h, ˜u_1) \\
M_2 &= (g, ˜g^2, ˜g^n, ˜h, ˜u_2) \\
& \vdots \\
M_n &= (g, ˜g^n, ˜g^n, ˜h, ˜u_n)
\end{align*}
\]

Each message \( M_i \) is \( (g, ˜g^i, ˜g^n, ˜h, ˜u_i) \) is signed using the mercurial signature scheme MSf, resulting in a signature \( σ_i \). The verification consists of checking the \( n \) message-signature pairs \((M_i, σ_i)\) using the prior mercurial signature Verifyf algorithm.

How might we form the glue element \( ˜h \)? As discussed in the introduction, it is important for the origin-hiding property that \( ˜h \) for a message \( m = (g, u_1, \ldots, u_n) \), where \( u_i = ˜g^{m_i} \), be a function of the \( m_i \)'s so that if another representative \( m' \in [m]_{R_m} \) gets signed, the corresponding \( M'_i \)'s are in the same equivalence classes as the original \( M_i \)'s for the original \( m \) (i.e., \( M'_i \in [M_i]_{R_m} \) for all \( 1 \leq i \leq n \)). Computing \( ˜h \) as \( ˜g^{R(m_1, \ldots, m_n)} \) for a random function \( R \) of the \( m_i \)'s would work, but how would the signer compute such a value? A pseudorandom function could be used instead, but it is not obvious how to compute it since the signer has the group elements \( u_1, \ldots, u_n \), but not their discrete logarithms \( m_1, \ldots, m_n \).

Our solution is as follows. Consider a polynomial \( p_m(x) \) parameterized by the \( m_i \)'s: \( p_m(x) = m_1 + m_2 x + m_3 x^2 + \cdots + m_n x^{n-1} \). The signer evaluates this polynomial at a secret value \( ˜x \) known only to him: \( p_m(˜x) \). The glue element could be computed by the signer as \( ˜h = ˜g^{p_m(˜x)} \); however, to ensure that it is pseudorandom, the signer picks a uniformly random \( w \leftarrow Z_p^* \) sets \( ˜g = g^w \), and computes the glue element as \( ˜h = ˜g^{p_m(˜x)} \). Additionally, the signer picks a uniformly random \( y \leftarrow Z_p^* \) and raises \( ˜g^{p_m(˜x)} \) to \( y \), resulting in the following:

\[
\tilde{h} = \left( \tilde{g}^{p_m(\tilde{x})} \right)^y = \left( \tilde{g}^{\sum_{i=1}^n m_\tilde{x}_i^{i-1}} \right)^y = \left( \prod_{i=1}^n g^m_i \right)^y \tag{1}
\]

Note that \( w \) is fresh for each signature, but \( y \) is the same for all signatures issued by the same signer. In reality, the signer does not know the \( m_i \)'s required to form the polynomial \( p_m(\tilde{x}) \); however, he is given as input the original \( u_i \)'s, which have the relationship \( u_i = ˜g^{m_i} \), so \( ˜h \) can be computed directly as follows, where \( \tilde{u}_i = u_i^w = ˜g^{m_i} \):

\[
\tilde{h} = \left( \prod_{i=1}^n \tilde{u}_i^{\tilde{x}_i^{i-1}} \right)^y \text{. This is exactly Equation (1).}
\]

We now describe our construction formally. We first provide a non-interactive construction that satisfies the input-output specification in the definition of mercurial signatures. The final construction (Section 4.1) involves an interactive signing protocol carried out between the signer and the recipient of the signature.

**Construction.** The following algorithms are invoked from the fixed-length mercurial signature scheme MSf: ChangeRepf(pk, m, σ, μ) → \((m', σ')\), where \( m' = μ^m \in [m]_{R_m} \) for a message converter \( μ \in Z_p^* \) and Verifyf(pk, m', σ') = 1, and ConvertSigf(pk, m, σ, ρ) → χ, where Verifyf(pk, m, χ) = 1 and pk = pkρ ∈ [pk]_{R_m} for a key converter ρ ∈ Z_p^*.

PPGenX(1k) → PPX: Run PP ← PPGenX(1k) and output PPX = \( (G_1, G_2, G_7, P, P, e) \).

KeyGenX(PPX) → \((pk_X, sk_X)\): Run \((pk, sk) ← KeyGen_f(PP, \ell = 5)\), where \( sk = (x_1, x_2, x_3, x_4, x_5) \in (Z_p^*)^5 \) and \( pk = (\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5) \in (G_2^*)^5 \) for \( \hat{X}_i = \hat{p}_x \). Pick uniformly at random a secret point
\[ \hat{x} \leftarrow Z_p^* \text{ and secret seeds } y_1, y_2 \leftarrow Z_p^* \text{. Also pick } x_6, x_8 \leftarrow Z_p^* \text{ and set } x_7 = x_6 \cdot \hat{x} \text{ and } x_9 = x_8 \cdot y_1 \text{ and } x_{10} = y_8 \cdot y_2. \text{ Set } sk_\mathbf{X} = (sk, x_6, x_7, x_8, x_9, x_{10}) \text{ and } pk_\mathbf{X} = (pk, x_6, \hat{X}_7, \hat{X}_8, \hat{X}_9, X_{10}) \text{, where } \hat{X}_i = \hat{P}^{x_i}, \text{ and output } (pk_\mathbf{X}, sk_\mathbf{X}). \]

**Sign\mathbf{X}(sk_\mathbf{X}, m) \rightarrow (h, \sigma):** On input \( sk_\mathbf{X} = (sk, x_6, x_7, x_8, x_9, x_{10}) \) and a message \( m = (g, u_1, \ldots, u_n) \in (\mathbb{G}_1)^n \), where \( g \) is a generator of \( \mathbb{G}_1 \), compute \( \hat{x} = x_7 \cdot x_6^{-1} \) and \( y_1 = x_9 \cdot x_8^{-1} \) and \( y_2 = x_{10} \cdot x_8^{-1} \). Then, compute \( y := y_1 \cdot y_2 \) and \( \hat{h} = (\prod_{i=1}^{n} u_i^{x_i^{-1}})^y \). Compute \( \hat{g}^2, \ldots, \hat{g}^n \). For all \( 1 \leq i \leq n \), form the message \( M_i = (\hat{g}, \hat{g}^i, \hat{g}^n, \hat{h}, u_i) \) and run \( \sigma_i \leftarrow \text{Sign}_f(sk, M_i) \). Output the signature \( (h, \sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}) \).

**Verify\mathbf{X}(pk_\mathbf{X}, m, (h, \sigma)) \rightarrow 0/1:** On input \( pk_\mathbf{X} = (pk, \hat{X}_6, \hat{X}_7, \hat{X}_8, \hat{X}_9, X_{10}) \), \( m = (g, u_1, \ldots, u_n) \), and a signature \( (h, \sigma = \{\sigma_1, \ldots, \sigma_n\}) \), compute \( g^{\hat{g}} \cdot g^{\hat{g}} \cdot \hat{g}^{\hat{n}} \). For all \( 1 \leq i \leq n \), form the message \( M_i = (g, \hat{g}^i, \hat{g}^n, \hat{h}, u_i) \) and check whether \( \text{Verify}_f(pk, M_i, \sigma_i) = 1 \). If these checks hold, output 1; otherwise output 0.

**ConvertSK\mathbf{X}(sk, \rho) \rightarrow \tilde{sk}_\mathbf{X}:** On input \( sk = (sk, x_6, x_7, x_8, x_9, x_{10}) \) and \( \rho \in Z_p^* \), run \( \tilde{sk} \leftarrow \text{ConvertSK}_f(sk, \rho) \), where \( \tilde{sk} = \rho \cdot x_6, \) compute \( \tilde{x}_i = \rho \cdot x_i \) for all \( 6 \leq i \leq 10 \), and output the new secret key \( \tilde{sk}_\mathbf{X} = (\tilde{sk}, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}) \).

**ConvertPK\mathbf{X}(pk_\mathbf{X}, \rho) \rightarrow \tilde{pk}_\mathbf{X}:** On input \( pk_\mathbf{X} = (pk, \hat{X}_6, \hat{X}_7, \hat{X}_8, \hat{X}_9, X_{10}) \) and \( \rho \in Z_p^* \), run \( \tilde{pk} \leftarrow \text{ConvertPK}_f(pk, \rho) \), where \( \tilde{pk} = pk^\rho \), compute \( \tilde{X}_i = \hat{X}_i^\rho \) for all \( 6 \leq i \leq 10 \), and output the new public key \( \tilde{pk}_\mathbf{X} = (pk, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8, \tilde{X}_9, \tilde{X}_{10}) \).

**ChangeRep\mathbf{X}(pk_\mathbf{X}, m, (h, \sigma), \mu) \rightarrow (m', (h', \sigma')):** On input \( pk_\mathbf{X} = (pk, \hat{X}_6, \hat{X}_7, \hat{X}_8, \hat{X}_9, X_{10}) \), \( m = (g, u_1, \ldots, u_n) \), \( (h, \sigma = \{\sigma_1, \ldots, \sigma_n\}) \), and \( \mu \in Z_p^* \), compute \( g^{\hat{g}^2}, \ldots, g^n \). For all \( 1 \leq i \leq n \), form the message \( M_i = (g, \hat{g}^i, \hat{g}^n, \hat{h}, u_i) \) and run \( \text{ChangeRep}(pk, M_i, \sigma_i, \mu) \). This is an interactive protocol between a Signer, who runs the Sign side of the protocol, and a Receiver, who runs the Receive side.

**4.1 Signing Protocol**

Our construction satisfies the input-output specification in the definition of mercurial signatures; however, unfortunately, our proofs of unforgeability and origin-hiding do not allow a signer to simply sign any message given to it as input. Instead, the signer must run a signing protocol with the receiver of the signature. When a signature is queried on a message \( m = (g, u_1, \ldots, u_n) \in (\mathbb{G}_1)^n \), the signer first has the recipient give a ZKPoK that, for all \( 1 \leq i \leq n \), the recipient knows \( m_i \) such that \( u_i = g^{m_i} \).

This ZKPoK is requisite for proving unforgeability, as the reduction’s algorithm must use the exponent \( m_i \)’s. The signer then carries out the signing algorithm \( \text{Sign}_\mathbf{X} \) as specified in the construction above, with one modification: the signer picks a uniformly random \( w \leftarrow Z_p^* \), sets \( \tilde{g} = g^w \), and computes the glue element \( \hat{h} \) relative to base \( g \). The additional randomness \( w \) ensures that the glue element is pseudorandom, as discussed in Section 4.

In addition to the usual unforgeability property that protects the signer, mercurial signatures also have the origin-hiding property that protects the privacy of the signature recipient. Intuitively, origin-hiding means that a message-signature pair \( (m, \sigma) \) is distributed exactly the same way whether (1) the signature \( \sigma \) on \( m \) was issued directly by the signer, or (2) \( (m, \sigma) \) was obtained by running \( \text{ChangeRep}(pk, m', \sigma') \) on an equivalent \( m' \).

The reason it protects the signature recipient is that the resulting \( (m, \sigma) \) is not linkable to the specific point in time when this recipient was issued this signature.

In order to satisfy the origin-hiding property, the glue element \( \hat{h} \) must be computed (relative to \( \tilde{g} \)) as a function of the entire equivalence class to which the message belongs. That way, no matter which message in the class is signed, the glue element’s discrete logarithm base \( \tilde{g} \) is the same. A dishonest signer might try to compute the glue element incorrectly, depriving the recipient of the benefits that origin-hiding confers. Thus, as a final step in the signing protocol, the signer verifies that the glue element was indeed computed correctly via a ZKPoK, so origin-hiding holds for all signers, not just honest ones.

**Signing Protocol:** This is an interactive protocol between a Signer, who runs the Sign side of the protocol, and a Receiver, who runs the Receive side.

\[
[\text{Sign}_\mathbf{X}(sk_\mathbf{X}, m) \leftrightarrow \text{Receive}_\mathbf{X}(pk_\mathbf{X}, m, (m_1, \ldots, m_n))] \rightarrow (m, (h, \sigma)) \text{: The Signer takes as input his signing key } sk_\mathbf{X} = (sk, x_6, x_7, x_8, x_9, x_{10}) \text{ and a message } m = (g, u_1, \ldots, u_n). \text{ The Receiver takes as input the corre-}
\]
While the elements $\tilde{\text{exp}}$, $\tilde{\text{group}}$, $\tilde{\text{mult}}$, $\tilde{\text{field}}$, $\tilde{\text{sign}}$, $\tilde{\text{verify}}$, $\tilde{\text{convert}}$, $\tilde{\text{convertSig}}$ are not part of the signature itself and a vector $(m_1, \ldots, m_n) \in (\mathbb{Z}_p^n)^n$.

0. The Receiver checks that in fact $u_i = \tilde{g}^{m_i}$ for all $1 \leq i \leq n$.

1. The Signer acts as the verifier while the Receiver gives a ZKPoK that, for all $1 \leq i \leq n$, he knows $m_i$ such that $u_i = \tilde{g}^{m_i}$. If the verification fails, the Signer denies the Receiver the signature.

2. The Signer computes $\tilde{h}$ as in the construction above. He then picks uniformly at random $w \leftarrow \mathbb{Z}_p^n$ and computes $\tilde{h} = \tilde{h}^w$ and $\tilde{m} = (\tilde{g}^w, u_1, \ldots, u_n) = (\tilde{g}^u, u_1, \ldots, u_n)$. He also computes $\tilde{g}^u, \tilde{g}^n$. For all $1 \leq i \leq n$, he forms the message $M_i = (\tilde{g}, \tilde{g}^n, \tilde{h}, u_i)$ and runs $\sigma_i \leftarrow \text{Sign}_f(\text{sk}, M_i)$. The Signer sends the message $\tilde{m}$ and signature $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to the Receiver.

3. The Receiver acts as the verifier while the Signer gives a ZKPoK that he has computed the glue element $\tilde{h}$ correctly. If verification of the glue and signature passes, the Receiver outputs the message $\tilde{m}$ and signature $(\tilde{h}, \sigma)$.

The algorithms $\text{Verify}_X$, $\text{ChangeRep}_X$, and $\text{ConvertSig}_X$ must be modified to take as input the message $\tilde{m} = (\tilde{g}, u_1, \ldots, u_n)$:

- $\text{Verify}_X(\text{pk}_X, \tilde{m}, (\tilde{h}, \sigma)) \rightarrow 0/1$: Form $M_i = (\tilde{g}, \tilde{g}^n, \tilde{h}, \tilde{u}_i)$ and check whether $\text{Verify}_f(\text{pk}, M_i, \sigma_i) = 1$ for all $1 \leq i \leq n$.

- $\text{ChangeRep}_X(\text{pk}_X, \tilde{m}, (\tilde{h}, \sigma), \mu) \rightarrow (\tilde{m}', (\tilde{h}', \sigma'))$: Form $M_i = (\tilde{g}, \tilde{g}^n, \tilde{h}, \tilde{u}_i)$ and run $(M'_i, \sigma'_i) \leftarrow \text{ChangeRep}_f(\text{pk}, M_i, \sigma_i, \mu)$ for all $1 \leq i \leq n$. Output $(\tilde{m}', (\tilde{h}', \sigma') = \{\sigma'_1, \ldots, \sigma'_n\})$.

- $\text{ConvertSig}_X(\text{pk}_X, \tilde{m}, (\tilde{h}, \sigma), \rho, \mu) \rightarrow (\tilde{m}', (\tilde{h}', \tilde{\sigma}))$: Run $(\tilde{m}', (\tilde{h}', \sigma') = \{\sigma'_1, \ldots, \sigma'_n\}) \leftarrow \text{ConvertSig}_f(\text{pk}, M'_i, \sigma'_i, \rho)$ for all $1 \leq i \leq n$. Output $(\tilde{m}', (\tilde{h}', \tilde{\sigma}) = \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n\})$.

Remark. While the elements $\tilde{x}_6 = \tilde{x}_6, \tilde{x}_7 = \tilde{x}_7, \tilde{x}_9 = \tilde{x}_9, \tilde{x}_{10} = \tilde{x}_{10}$ of the public key are not used in signature verification, they are used in Step 3 of the signing protocol. The secret values $\tilde{x}, \tilde{y}_1, \tilde{y}_2$ are defined relative to random bases in order for the hiding proofs to go through. The secret value $\tilde{y}$ is broken into two components, $y_1$ and $y_2$, in order for the proof of unforgeability to go through (specifically, Claim 3).

![Fig. 1. Table of efficiency for MSX. Here, exp denotes the number of group exponentiations, mult denotes the number of group multiplications, and pair denotes the number of pairings. Field operations are ignored. Group and field elements, grp and field, are given as the total number of elements output.](image)

Correct verification and key conversion can be seen by inspection. We show correct change of message representative, and signature conversion is similar.

Change of message representative: We wish to show that for all messages $m \in M_n$, for all signatures $(\tilde{h}, \sigma)$ such that $\text{Verify}_X(\text{pk}_X, \tilde{m}, (\tilde{h}, \sigma)) = 1$, for all $\mu \in \text{sample}_\mu$, for all $(\tilde{m}', (\tilde{h}', \sigma')) \in \text{ChangeRep}_X(\text{pk}_X, \tilde{m}, (\tilde{h}, \sigma), \mu)$, it holds that $\text{Verify}_X(\text{pk}_X, \tilde{m}', (\tilde{h}', \sigma')) = 1$, where $\tilde{m}' \in [\tilde{m}]_\text{rep}$. First, observe that the $M_i$'s corresponding to $(\tilde{m}, (\tilde{h}, \sigma) = \{\sigma_1, \ldots, \sigma_n\})$ are $M_i = (\tilde{g}, \tilde{g}^n, \tilde{h}, \tilde{u}_i)$. ChangeRep invokes ChangeRep$_f$ as follows: for all $1 \leq i \leq n$, ChangeRep$_f(\text{pk}, M_i, \sigma_i, \mu)$ outputs $M'_i, \sigma'_i$, where $M'_i = (\tilde{g}^i, \tilde{g}^n_i, \tilde{h}^n, \tilde{u}_i)$. By correct change of message representative of ChangeRep$_f$ (Theorem 1), we have that $\text{Verify}_f(\text{pk}, M'_i, \sigma'_i) = 1$ for all $1 \leq i \leq n$, which implies that $\text{Verify}_X(\text{pk}_X, \tilde{m}', (\tilde{h}', \sigma')) = 1$, where $\tilde{m}' = (\tilde{g}^i, \tilde{u}_i, \ldots, \tilde{u}_n) \in [\tilde{m}]_\text{rep}$.  

4.2 Origin-hiding

**Theorem 3 (Origin-hiding).** Let $\text{MS}_f$ be a mercurial signature scheme on message space $(\mathbb{G}_1)^5$ as in Theorem 1, and let $\text{MS}_X$ be the variable-length mercurial signature scheme on message space $(\mathbb{G}_1)^n+1$ constructed above, where all signatures are issued via the interactive signing protocol. Then, $\text{MS}_X$ is correct.
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rem 1, and let MSX be the variable-length mercurial signature scheme on message space \( (G_1^*)^{n+1} \) constructed above. Suppose all signatures are issued via the interactive signing protocol described in Section 4.1, where the proof system used in Step 3 is sound under sequential (or concurrent) composition. Then, MSX is origin-hiding under sequential (or concurrent) composition.

Origin-hiding of ChangeRepX: Let \( \text{pk}_X, \tilde{m}, (\bar{h}, \sigma) = (\sigma_1, \ldots, \sigma_n) \) be such that \( \text{Verify}_X(\text{pk}_X, \tilde{m}, (\bar{h}, \sigma)) = 1 \), where \( \text{pk}_X \) is possibly adversarially generated. \( \text{ChangeRep}_X(\text{pk}_X, \tilde{m}, (\bar{h}, \sigma), \mu) \) outputs \((\tilde{m}', (\bar{h}', \bar{\sigma}')) = (\tilde{m}^\mu, (\bar{h}^\mu, (\sigma_1', \ldots, \sigma_n'))) \), where \( \tilde{m}^\mu \) is shorthand for \( \tilde{m} = (\bar{g}^\mu, \bar{u}_1^\mu, \ldots, \bar{u}_n^\mu) \). By soundness of the ZKPoK in Step 3 of the signing protocol, the glue element \( \bar{h} \) is computed correctly with overwhelming probability. The \( M_i \)'s corresponding to \((\tilde{m}, (\bar{h}, \sigma))\) are \( M_i = (\bar{g}, \bar{g}^\mu, \bar{g}^\sigma, \bar{h}, \bar{u}_i) \). \( \text{ChangeRep}_X \) invokes \( \text{ChangeRep}_X \) as follows: for all \( 1 \leq i \leq n \), \( \text{ChangeRep}_X(\text{pk}, M_i, (\sigma_i'), \mu) \) outputs \((M_i', \sigma_i')\), where \( M_i' = (\bar{g}^\mu, (\bar{g}^\sigma)^i, (\bar{g}^\sigma)^n, \bar{h}^\mu, \bar{u}_i^\mu) \). By origin-hiding of \( \text{ChangeRep}_X \) (Theorem 1), \( \sigma_i' \) is distributed the same as a fresh signature on \( M_i' \) for all \( 1 \leq i \leq n \). Note that the glue element \( \bar{h}^\mu \) is correct if \( \bar{h} \) is correct, and \( \bar{h}^{\mu} \) is distributed the same as a fresh glue element for a fresh signature on \( \bar{h}^{\mu} \). Thus, \( \tilde{m}^\mu \) is a uniformly random element of \( [\tilde{m}]_{\mathbb{R}_m} \), and \((\tilde{m}^\mu, (\sigma_1', \ldots, \sigma_n'))\) is a uniformly random element in the space of signatures \((\bar{h}, \tilde{\sigma})\) satisfying \( \text{Verify}_X(\text{pk}_X, \tilde{m}^\mu, (\bar{h}, \tilde{\sigma})) = 1 \) with overwhelming probability.

Origin-hiding of ConvertSigX: Let \( \text{pk}_X, \tilde{m}, (\bar{h}, \sigma) = (\sigma_1, \ldots, \sigma_n) \) be such that \( \text{Verify}_X(\text{pk}_X, \tilde{m}, (\bar{h}, \sigma)) = 1 \), where \( \text{pk}_X \) is possibly adversarially generated. \( \text{ConvertSig}_X(\text{pk}_X, \tilde{m}, (\bar{h}, \sigma), \rho, \mu) \) outputs \((\tilde{m}', (\bar{h}', \bar{\sigma}')) = (\tilde{m}^\mu, (\bar{h}^\mu, (\sigma_1', \ldots, \sigma_n'))) \), where \( \tilde{m}^\mu \) is shorthand for \( \tilde{m} = (\bar{g}^\mu, \bar{g}^\rho, \bar{u}_1^\mu, \ldots, \bar{u}_n^\mu) \). By soundness of the ZKPoK in Step 3 of the signing protocol, the glue element \( \bar{h} \) is computed correctly with overwhelming probability. The \( M_i \)'s corresponding to \((\tilde{m}, (\bar{h}, \sigma))\) are \( M_i = (\bar{g}, \bar{g}^\mu, \bar{g}^\rho, \bar{h}, \bar{u}_i) \). The output of \( \text{ConvertSig}_X \) is computed in two steps. First, \( \text{ChangeRep}_X(\text{pk}_X, \tilde{m}, (\bar{h}, \sigma), \mu) \) outputs \((\tilde{m}', (\bar{h}', \bar{\sigma}')) = (\tilde{m}^\mu, (\bar{h}^\mu, (\sigma_1', \ldots, \sigma_n'))) \). Then, \( \text{ConvertSig}_X(\text{pk}_X, M_i, (\sigma_i', \rho), \mu) \) outputs \( \sigma_i \) for all \( 1 \leq i \leq n \). \( \text{ChangeRep}_X \) is origin-hiding, as shown above, and \( \text{ConvertSig}_X \) is origin-hiding by Theorem 1. The glue element \( \tilde{m} \) is a uniformly random element of \( [\tilde{m}]_{\mathbb{R}_m} \), and \((\bar{h}', \sigma_1', \ldots, \sigma_n')\) is a uniformly random element in the space of signatures \((\bar{h}, \tilde{\sigma})\) satisfying \( \text{Verify}_X(\text{pk}_X, \tilde{m}, (\bar{h}, \tilde{\sigma})) = 1 \) with overwhelming probability (where \( \text{ConvertPK}_X(\text{pk}_X, \rho) = (\text{pk}_X)^\rho \) is a uniformly random element of \( [\text{pk}_X]_{\mathbb{R}_m} \)). Note that origin-hiding does not hold if \( \bar{h} = 1 \), but this occurs with negligible probability.

4.3 Unforgeability

Unforgeability of MSX holds under a variant of the asymmetric bilinear decisional Diffie-Hellman assumption (ABDDH+) introduced by Fuchsbauer et al. [21].

Definition 9 (ABDDH+ assumption [21]). Let BG be a bilinear group generator that outputs \( \text{BG} = (p, G_1, G_2, G_T, P, \bar{P}, e) \). The ABDDH+ assumption holds in \( G_1 \) if for all probabilistic, polynomial-time (PPT) algorithms \( A \), there exists a negligible function \( \nu \) such that:

\[
\Pr[b \leftarrow \{0, 1\}; \text{BG} \leftarrow \text{BGGen}(1^k); u, v, w, r \leftarrow \mathbb{Z}_p^*; \\
b^* \leftarrow A(\text{BG}, \hat{P} u, \hat{T} v, P u, \hat{T} w, P (1-b)-r+b \cdot (u w^e)) \\
: b^* = b - \frac{1}{2} \leq \nu(k)
\]

Proposition 1. [21] The ABDDH+ assumption holds in generic groups.

Theorem 4 (Unforgeability). Let MSX be a mercurial signature scheme on message space \( (G_1^*)^5 \) as in Theorem 1, and let MSX be the variable-length mercurial signature scheme on message space \( (G_1^*)^{n+1} \) constructed above. Suppose all signatures are issued via the interactive signing protocol described in Section 4.1, where the proof system used in Step 1 is extractable under sequential (or concurrent) composition. Then, unforgeability of MSX holds sequentially (or concurrently) under the discrete logarithm (DL) assumption in \( G_2 \) and the ABDDH+ assumption in \( G_1 \). The same holds when \( G_1 \) and \( G_2 \) are swapped.

Proof. We wish to show that if there exists a probabilistic, polynomial-time (PPT) adversary \( A \) that breaks unforgeability of MSX with non-negligible probability, then we can construct a PPT adversary \( \mathcal{A}' \) that breaks unforgeability of MSX with non-negligible probability, or the discrete logarithm (DL) or ABDDH+ assumption doesn’t hold.

Suppose there exists such a PPT adversary \( A \). Then, we construct a PPT adversary \( \mathcal{A}' \) as a reduction \( \mathcal{B}_{MSX} \) running \( A \) as a subroutine. We construct the reduction \( \mathcal{B}_{MSX} \) for breaking unforgeability of MSX as follows. \( \mathcal{B}_{MSX} \) receives as input public parameters \( PP = \text{BG} = (G_1, G_2, G_T, P, \bar{P}, e) \) and a fixed public key \( pk = (\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5) \) for the mercurial signature scheme MSX on messages of length \( \ell = 5 \) for
which he will try to produce a forgery. He chooses uniformly at random a secret point $\hat{x} \leftarrow \mathbb{Z}_p^*$ and secret seeds $y_1, y_2 \leftarrow \mathbb{Z}_p^*$. He also picks $x_6, x_8 \leftarrow \mathbb{Z}_p^*$ and sets $x_7 = x_6 \cdot \hat{x}$ and $x_9 = x_8 \cdot y_1$ and $x_{10} = x_7 \cdot y_2$. He then sets $pk_X = (pk, X_0, X_1, X_2, X_3, X_4, X_5)$, where $X_i = \hat{P}^{x_i}$. $B_{MS_f}$ forwards $PP_X = PP$ and $pk_X$ to $A$ and acts as $A$’s challenger $C$. As in the unforgeability game for $MS_f$, $B_{MS_f}$ has access to a signing oracle $Sign_f(sk, \cdot)$, where $sk$ is the secret key corresponding to $pk$. $A$ proceeds to make signature queries on messages of the form $m = (\hat{g}, u_1, \ldots, u_n) \in (\mathbb{Z}_p^*)^{n+1}$. For each signature query, $B_{MS_f}$ acts as the verifier while $A$ gives a ZKPoK that, for all $1 \leq i \leq n$, he knows $m_i$ such that $u_i = \hat{g}^{m_i}$. If the verification fails, $B_{MS_f}$ denies $A$ the signature; otherwise, $B_{MS_f}$ computes $y = y_1 \cdot y_2$ and $\hat{h} = \left( \prod_{i=1}^{n} u_i^{\hat{x}_i - 1} \right)^y$. $B_{MS_f}$ picks uniformly at random $w \leftarrow \mathbb{Z}_p^*$ and computes $\tilde{g} = \hat{g}^w, \hat{h} = \tilde{h}^w$, and $\tilde{u}_i = u_i^w$ for all $1 \leq i \leq n$. He also computes $\tilde{g}^2, \ldots, \tilde{g}^n$. He forwards $n$ messages of the form $M_i = (\tilde{g}, \tilde{g}^i, \tilde{g}^n, \tilde{u}_i)$ to his signing oracle $Sign_f(sk, \cdot)$ and receives $n$ signatures $\sigma_1, \ldots, \sigma_n$. He sends the message $\hat{m} = (\tilde{g}, \tilde{u}_1, \ldots, \tilde{u}_n)$ and the signature $(\hat{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a ZKPoK that $\hat{h}$ was computed correctly.

After some polynomial number of signature queries, $A$ produces a forgery $(pk_{X_f}, \hat{m}^*, (\hat{h}^*, \sigma^*))$, where $pk_{X_f} = (pk^*, X_0^*, X_1^*, X_2^*, X_3^*, X_4^*, X_5^*)$, $\hat{m}^* = (\tilde{g}^*, \tilde{u}_1^*, \ldots, \tilde{u}_n^*)$, and $\sigma^* = \{\sigma_1^*, \ldots, \sigma_n^*\}$. $A$’s forgery can be represented as a set of messages that are in the message space of $MS_f$: $M_1^* = (\tilde{g}^*, (\tilde{g}^*)^1, (\tilde{g}^*)^n, \hat{h}^*, \tilde{u}_1^*)$, $M_2^* = (\tilde{g}^*, (\tilde{g}^*)^2, (\tilde{g}^*)^n, \hat{h}^*, \tilde{u}_2^*)$, ..., $M_n^* = (\tilde{g}^*, (\tilde{g}^*)^n, (\tilde{g}^*)^n, \hat{h}^*, \tilde{u}_n^*)$. $B_{MS_f}$ chooses $i \leftarrow \{1, \ldots, n\}$ uniformly at random and outputs $(pk_{X_f}, M_i^*, \sigma_i^*)$ as his forgery. Let us analyze $B_{MS_f}$’s success probability.

Suppose $A$’s forgery $(pk_{X_f}, \hat{m}^*, (\hat{h}^*, \sigma^*))$ is successful. Then, by definition, it satisfies $[pk_{X_f}]_R = [pk]_R$ and $\forall \hat{m} \in Q, [\hat{m}^*]_R \neq [m]_R$ and $Verify_X(pk_{X_f}, \hat{m}^*, (\hat{h}^*, \sigma^*)) = 1$, where $Q$ is the set of discrete logarithms $\hat{m} = \{m_1, \ldots, m_n\} \in (\mathbb{Z}_p^*)^n$. $A$ has queried to the signing oracle. Note that the forged $\tilde{g}^*$ and $\hat{h}^*$ must be repeated for each message $M_i^*$ because the verification algorithm accepts the signature $(\hat{h}^*, \sigma^*) = \{\sigma_1^*, \ldots, \sigma_n^*\}$.

There are two ways in which the forged message $\hat{m}^*$ could have been derived by $A$:

1. **Good Case:** There exists some $i \in \{1, \ldots, n\}$ for which $[M_i^*]_R \neq [M]_R$ for any $M$ previously queried by $B_{MS_f}$ to his signing oracle. We will see that with overwhelming probability, the Good Case is the way in which $A$ forms his forgery.

2. **Bad Case:** Every $M_i^*$ is such that $[M_i^*]_R = [M]_R$ for some $M$ previously queried by $B_{MS_f}$ to his signing oracle. In this case, $A$ is able to ‘mix and match* $m_i$’s from different messages for which signatures have been issued. We claim that $A$ cannot do this, except with negligible probability, or the DL or ABDDH+ assumption doesn’t hold.

First, note that if a glue element $\tilde{h}$ is formed as $\tilde{g}^{R(m_1, \ldots, m_n)}$ for some random function $R : (\mathbb{Z}_p^*)^n \rightarrow \mathbb{Z}_p^{*}$, then $A$ cannot mix and match. This is because if the vectors $(m_1, \ldots, m_n)$ are distinct, then the values $R(m_1, \ldots, m_n)$ are distinct as well as the glue elements $\tilde{g}^{R(m_1, \ldots, m_n)}$. Our goal is to demonstrate that a glue element formed as $\tilde{g}^{R(m_1, \ldots, m_n)}$ is indistinguishable from a real glue element $\tilde{g}^q$, where $q = p(\hat{x}) = \sum_{i=1}^{n} m_i \hat{x}_i - 1$. Then, $A$ can’t mix and match when real glue elements are used, except with negligible probability.

We achieve this goal in two steps. We first demonstrate that $\tilde{g}^{R(m_1, \ldots, m_n)}$ is indistinguishable from $\tilde{g}^R(q)$, where $R : \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p$ is a random function, under the DL assumption. We then demonstrate that $\tilde{g}^R(q)$ is indistinguishable from a real glue element $\tilde{g}^q$ under the ABDDH+ assumption. This gives the desired result.

Consider the following set of games. In Game 0, the real signing game, the glue element is computed directly, without extraction of the $m_i$’s or simulated proofs. Game 1 includes simulated proofs. In Games 2-5, the challenger acts as the zero-knowledge extractor to extract the $m_i$’s necessary to compute the glue element and provides a simulated proof that it was computed correctly. The overall proof structure is as follows. Arrows indicate why consecutive games are indistinguishable.

Game 0. $\tilde{h} = \tilde{g}^q$. No extraction or simulation. This is the real signing game.

\[ \$ \] Claim 1: zero-knowledge property

Game 1. $\tilde{h} = \tilde{g}^q$. No extraction, but simulation.

\[ \$ \] Claim 2: knowledge extractor property

Game 2. $\tilde{h} = \tilde{g}^q$, $q = p(\hat{x})$. Extraction and simulation henceforth.

\[ \$ \] Claim 3: ABDDH+ assumption in $G_1$

Game 3. $\tilde{h} = \tilde{g}^R(q)$, $q = p(\hat{x})$, $R : \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p$ random.

\[ \$ \] Claim 4: polynomial collision argument / Claim 5: DL assumption in $G_2$

Game 4. $\tilde{h} = \tilde{g}^R(q)$, $q = p(\alpha)$, $\alpha$ is a secret element in $\mathbb{Z}_p^*$.

\[ \$ \] Claim 6: polynomial collision argument

Game 5. $\tilde{h} = \tilde{g}^{R(m_1, \ldots, m_n)}$, $R : (\mathbb{Z}_p^*)^n \rightarrow \mathbb{Z}_p$ random.

We now provide descriptions of the games and proofs of the claims.
Game 0. In this real signing game, the glue element is $\tilde{h} = \tilde{g}^{y^q}$. There is no extraction or zero-knowledge simulation.

The challenger $C$ computes the public parameters $PP$ and keys $(pk, sk) = ((\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4, \tilde{X}_5), (x_1, x_2, x_3, x_4, x_5))$ for a mercurial signature scheme $MS_f$ on messages of length $\ell = 5$. $C$ chooses uniformly at random a secret point $\tilde{x} \leftarrow \mathbb{Z}_p^*$ and secret seeds $y_1, y_2 \leftarrow \mathbb{Z}_p^*$. He also picks $x_6, x_8 \leftarrow \mathbb{Z}_p^*$ and sets $x_7 = x_6 \cdot \tilde{x}$ and $x_9 = x_8 \cdot y_1$ and $x_{10} = x_8 \cdot y_2$. He then sets $pk_C = (pk, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8, \tilde{X}_9, \tilde{X}_{10})$, where $\tilde{X}_i = \tilde{p}^{x_i}$. $C$ forwards $PP_C$ and $pk_C$ to $A$.

$A$ proceeds to make signature queries on messages of the form $m = (\tilde{g}, u_1, \ldots, u_n) \in (\mathbb{G}_1^*)^{n+1}$. For each signature query, $C$ acts as the verifier while $A$ gives a $\text{ZKPoK}$ that, for all $1 \leq i \leq n$, he knows $m_i$ such that $u_i = \tilde{g}^{m_i}$. If the verification fails, $C$ denies $A$ the signature; otherwise, $C$ computes $y = y_1 \cdot y_2$ and $\tilde{h} = (\prod_{i=1}^n u_i^{x_{i-1}})^y$. $C$ picks uniformly at random $w \leftarrow \mathbb{Z}_p^*$ and computes $\tilde{g} = \tilde{g}^w, \tilde{h} = \tilde{h}^w, \tilde{u}_i = u_i^w \forall i$. He also computes $\tilde{g}^2, \ldots, \tilde{g}^n$. He then signs $m$ messages of the form $M_i = (\tilde{g}, \tilde{g}^i, \tilde{h}, \tilde{u}_i)$ using his secret key $sk$ for $MS_f$ and sends $m = (\tilde{g}, u_1, \ldots, u_n)$ and $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a simulated $\text{ZKPoK}$ that $\tilde{h}$ was computed correctly. $A$ issues queries for signatures on messages a polynomial number of times. The game ends when $A$ produces a forgery or terminates without producing a forgery.

Game 1. In this game, the glue element is $\tilde{h} = \tilde{g}^{y_q}$. There is no extraction, but now there is simulation.

Game 1 is the same as Game 0, except the challenger $C$ simulates the $\text{ZKPoK}$ that $\tilde{h}$ was computed correctly.

Claim 1. A PPT adversary cannot distinguish Game 0 from Game 1, except with negligible probability.

The only difference between the two games is zero-knowledge simulation. In Game 1, the challenger simulates the $\text{ZKPoK}$ that the glue $\tilde{h}$ was computed correctly, whereas in Game 0, the challenger gives a real $\text{ZKPoK}$. If an adversary could distinguish the two games, it would break the zero-knowledge property.

Game 2. In this game, the glue element is $\tilde{h} = \tilde{g}^{y_q}$, where $q = p(x)$, $g = p(x)$, $R : \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p^*$ is a random function.

Game 2 is the same as Game 2, except the challenger $C$ chooses a random function $R : \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p^*$ for each signature $h = \tilde{g}^{R(q)}$, where $q = p(\tilde{x})$. The rest of the signing protocol is carried out as in Game 2.
Claim 3. A PPT adversary cannot distinguish Game 2 from Game 3 under the ABDDH* assumption in $G_1$.

Consider the following decisional problem related to the ABDDH* assumption.

Definition 10. (ABDDH* problem). Let $BGGen$ be a bilinear group generator that outputs $BG = (G_1, G_2, G_T, P, \hat{P}, e)$. The ABDDH* problem in $G_1$ is to distinguish between the distributions $D_0$ and $D_1$ defined by:

$$D_0 = (BG \leftarrow BGGen(1^k); \alpha, \beta, u, v, \omega \leftarrow Z_p^*)$$

$$D_1 = (BG \leftarrow BGGen(1^k); \alpha, \beta, u, v, \omega \leftarrow Z_p^*)$$

Lemma 1. If the ABDDH* assumption holds for a bilinear group generator $BGGen$, then the ABDDH* problem is also hard for $BGGen$.

Indeed, a reduction $B$ given an ABDDH* instance

$$(BG, \hat{P}_u, \hat{P}_v, P_{au}, P_{av}, P_{uv}, P_{(1-b)-r+b(\omega uv)})$$

can pick uniformly at random $\alpha, \beta \leftarrow Z_p^*$ and provide an ABDDH* instance

$$(BG, \hat{P}_u, \hat{P}_v, P_{(1-b)-r+b(\omega uv)})$$

to an adversary $A$ whose non-negligible advantage in distinguishing ABDDH* tuples becomes $B$'s non-negligible advantage in breaking ABDDH*.

We now prove Claim 3 via a hybrid argument. Let $\Gamma(k)$ be a polynomial. For $0 \leq i \leq \Gamma(k)$, let $H_i$ be the hybrid experiment defined as the following game. The challenger $C$ computes the public parameters $PP$ and keys $(pk, sk) = ((X_1, X_2, X_3, X_4, X_5), (x_1, x_2, x_3, x_4, x_5))$ for a mercurial signature scheme $MS_f$ on messages of length $\ell = 5$. $C$ chooses uniformly at random a secret point $\hat{x} \leftarrow Z_p^*$ and secret seeds $y_1, y_2 \leftarrow Z_p^*$. He also picks $x_6, x_8 \leftarrow Z_p^*$ and sets $x_7 = x_6 \cdot \hat{x}$ and $x_9 = x_8 \cdot y_1$ and $x_{10} = x_8 \cdot y_2$. He then sets $pk_x = (pk, \hat{X}_6, \hat{X}_7, \hat{X}_9, \hat{X}_{10})$, where $\hat{X}_i = \hat{P}x_i$. $C$ also chooses a random function $R : Z_p^* \rightarrow Z_p^*$ and forwards $PP_X$ and $pk_x$ to $A$.

Let $A$’s $j$th signature query be on message $m_j = (\hat{y}_j, u_{j,1}, \ldots, u_{j,n})$. $C$ acts as the extractor while $A$ gives a ZKPoK that, for all $1 \leq i \leq n$, he knows $m_{j,i}$ such that $u_{j,i} = \hat{y}_j^{m_{j,i}}$. $C$ extracts the $m_{j,i}$’s, or if the extraction fails, $C$ denies $A$ the signature. Otherwise, $C$ computes the polynomial $p_j(x) = m_{j,1}x + m_{j,2}x^2 + \cdots + m_{j,n}x^{n-1}$ and evaluates $q_j = p_j(\hat{x})$. $C$ also computes $y = y_1 \cdot y_2$.

1. If $j \leq i$, $C$ computes $R(y_j)$ and $\hat{h}_j = \hat{g}_j^{R(y_j)}$. $C$ picks uniformly at random $w_j \leftarrow Z_p^*$ and computes $\tilde{g}_j = \hat{g}_j^{w_j}$, $\tilde{h}_j = \hat{h}_j^{w_j}$, and $\tilde{u}_{j,i} = u_{j,i}^{w_j \cdot y_i}$. He also computes $\tilde{g}_j^2, \ldots, \tilde{g}_j^n$. $C$ then signs $n$ messages of the form $M_{j,i} = (\hat{y}_j, \tilde{g}_j^i, \tilde{h}_j, \tilde{u}_{j,i})$ using his secret key sk for $MS_f$ and sends $m_j = (\tilde{g}_j, \tilde{u}_{j,1}, \ldots, \tilde{u}_{j,n})$ and $(\hat{h}_j, \sigma_j = \{\sigma_{j,1}, \ldots, \sigma_{j,n}\})$ to $A$, along with a simulated ZKPoK that $\hat{h}_j$ was computed correctly.

2. If $j > i$, $C$ computes $\hat{h}_j = \hat{g}_j^{w_j}$. $C$ picks uniformly at random $w_j \leftarrow Z_p^*$ and computes $\tilde{g}_j = \hat{g}_j^{w_j}$, $\tilde{h}_j = \hat{h}_j^{w_j}$, and $\tilde{u}_{j,i} = u_{j,i}^{w_j \cdot y_i}$. He also computes $\tilde{g}_j^2, \tilde{g}_j^n$. $C$ then signs $n$ messages of the form $M_{j,i} = (\tilde{g}_j, \tilde{g}_j^i, \tilde{h}_j, \tilde{u}_{j,i})$ using his secret key sk for $MS_f$ and sends $m_j = (\tilde{g}_j, \tilde{u}_{j,1}, \ldots, \tilde{u}_{j,n})$ and $(\hat{h}_j, \sigma_j = \{\sigma_{j,1}, \ldots, \sigma_{j,n}\})$ to $A$, along with a simulated ZKPoK that $\hat{h}_j$ was computed correctly.

By definition, $H_0$ corresponds to the game in which all glue elements are formed as $\tilde{h}_j = \hat{g}_j^{R(y_j)}$ (Game 2), while $H_{\Gamma(k)}$ corresponds to the game in which all glue elements are formed as $\tilde{h}_j = \hat{g}_j^{R(y_j)}$ (Game 3).

Let $A$ be an adversary, let $\Gamma(k)$ be the number of queries $A$ makes, and let $0 \leq i \leq \Gamma(k) - 1$. We wish to show that $A$’s advantage $\epsilon = Adv(A, k, i)$ in distinguishing $H_i$ from $H_{i+1}$ is negligible; in fact, $\epsilon < \nu$, where $\nu$ is the best advantage in distinguishing ABDDH* tuples.

Suppose not; that is, suppose $\epsilon = Adv(A, k, i) > \nu$ for some $A, k, i$. Then, let us show that there exists a probabilistic, polynomial-time $B$ that can distinguish between the distributions $D_0$ and $D_1$.

We construct $B$ as a reduction running $A$ as a subroutine. $B$ serves as the challenger for $A$ in the hybrid game and as the adversary for his own challenger in the ABDDH* game. $B$ receives as input $(BG, \hat{A}_0, \hat{A}_1, A_2, B_1, C, B_2, D)$, where implicitly $\hat{A}_0 = \hat{P}^\alpha, \hat{A}_1 = \hat{P}^\alpha u, \hat{A}_2 = \hat{P}^\alpha u v, B_1 = P^\beta, C = P^\beta u v, B_2 = P^\nu, D = D^\nu$ or $P^\nu$ for some uniformly random $\alpha, \beta, u, v, r, \omega \in Z_p^*$.

$B$ computes the public parameters $PP$ and keys $(pk, sk) = ((\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5), (x_1, x_2, x_3, x_4, x_5))$ for a mercurial signature scheme $MS_f$ on messages of length $\ell = 5$. $B$ chooses uniformly at random a secret point $\hat{x} \leftarrow Z_p^*$ but does not know the secret seeds $y_1, y_2$. He also picks $x_6 \leftarrow Z_p^*$ and sets $x_7 = x_6 \cdot \hat{x}$. He then sets $pk_x = (pk, \hat{X}_6, \hat{X}_7, \hat{X}_9, \hat{X}_{10})$, where $\hat{X}_i = \hat{P}x_i$. $B$ also chooses a random function $R : Z_p^* \rightarrow Z_p^*$ and forwards $PP_X$ and $pk_x$ to $A$.

$A$ proceeds to make queries to the signing oracle. Acting as the challenger for $A$, $B$ is responsible for computing the responses to the signature queries and forwarding them to $A$. $B$ responds to the signature queries as follows.
Let A’s jth signature query be on message \( m_j = (\hat{y}_j, u_{j,1}, \ldots, u_{j,n}) \). B acts as the extractor while A gives a ZKPoK that, for all \( 1 \leq i \leq n \), he knows \( m_j, j \) such that \( u_{j,i} = \hat{g}^n_j \). B extracts the \( m_j, j \)'s, or if the extraction fails, B denies A the signature. Otherwise, B computes the polynomial \( p_j(x) = m_{j,1} + m_{j,2}x + \cdots + m_{j,n}x^{n-1} \) and evaluates \( \hat{y}_j = p_j(\hat{x}) \).

(1) If \( j \leq i \), B computes \( R(\hat{y}_j) \) and \( \tilde{h}_j = \hat{g}^{R(\hat{y}_j)}_j \). B picks uniformly at random \( w_j \leftarrow \mathbb{Z}_p^* \) and computes \( \tilde{g}_j = \tilde{g}_j^{w_j}, \tilde{h}_j = \tilde{h}_j^{w_j}, \) and \( \tilde{u}_{j,i} = u_{j,i}^{w_j} \). He also computes \( \tilde{g}_j^{w_j}, \tilde{g}_j^n \). He then signs \( n \) messages of the form \( M_{j,i} = (\hat{y}_j, \hat{g}_j, \tilde{g}_j^n, \tilde{h}_j, \tilde{u}_{j,i}) \) using his secret key sk for MSF and sends \( m_j = (\hat{y}_j, \hat{g}_j, \tilde{g}_j^n, \tilde{h}_j, \tilde{u}_{j,i}, \tilde{t}_{j,i}) \) to A, along with a simulated ZKPoK that \( \tilde{h}_j \) was computed correctly.

(2) If \( j > i \), B computes \( \tilde{h}_j = D^{q_j} \). B sets \( \hat{y}_j = B_2^{q_j}, \tilde{h}_j = \tilde{h}_j, \) and \( \tilde{u}_{j,i} = B_2^{q_j,i} \). He also computes \( B_2, B_3, B_2^\prime \). He then signs \( n \) messages of the form \( M_{j,i} = (B_2, B_3, B_2^\prime, D^{q_j}, B_2^\prime) \) using his secret key sk for MSF and sends \( m_j = (B_2, B_3, B_2^\prime, D^{q_j}, B_2^\prime) \) and \( (D^{q_j}, \tilde{u}_{j,i} = \{q_j, 1, \ldots, q_j, n\}) \) to A, along with a simulated ZKPoK that \( \tilde{h}_j \) was computed correctly.

(3) If \( j > i + 1 \), B computes: \( \tilde{h}_j = C^{q_j} \). B picks uniformly at random \( w_j \leftarrow \mathbb{Z}_p^* \) and computes \( \tilde{g}_j = B_2^{q_j}, \tilde{h}_j = \tilde{h}_j^{w_j}, \) and \( \tilde{u}_{j,i} = (B_2^{q_j,i})^{w_j} \). He also computes \( (B_2^{q_j,i})^2, \ldots, (B_2^{q_j,i})^n \). He then signs \( n \) messages of the form \( M_{j,i} = (B_2^{q_j,i}, (B_2^{q_j,i})^2, (B_2^{q_j,i})^n, C^{q_j}, (B_2^{q_j,i})^{m,j} \) using his secret key sk for MSF and sends \( m_j = (B_2^{q_j,i}, (B_2^{q_j,i})^2, (B_2^{q_j,i})^n, C^{q_j}, \tilde{u}_{j,i} = \{q_j, 1, \ldots, q_j, n\}) \) to A, along with a simulated ZKPoK that \( \tilde{h}_j \) was computed correctly.

Finally, when A terminates, without loss of generality he outputs either 0 or 1. He outputs 0 if he thinks he has observed \( H_k \) and 1 if he thinks he has observed \( H_{k+1} \). If A outputs 0, B outputs 0; otherwise, B outputs 1. Let us analyze B’s success probability.

First, note that in the public key pk, the values \( \hat{x}_8, \hat{x}_9, \hat{x}_{10} \) can’t be computed as \( \hat{x}_8 = \hat{x}_9 = \hat{x}_{10} \) due to \( x_9 = x_8 + y_1, x_10 = x_9 + y_2 \), where B does not know \( y_1 \) or \( y_2 \); however, \( \hat{x}_0, \hat{x}_1, \hat{x}_2 \) is implicitly \( \hat{x}_0 = \hat{x}_1 = \hat{x}_2 \). A also computes \( R(\hat{y}_j) \) for uniformly random \( x_8, y_1, y_2 \in \mathbb{Z}_p^* \). Thus, \( pk \) is distributed correctly.

The case \( j < i \) is exactly as in the hybrid game. For the case \( j > i + 1 \), B1 is implicitly \( \hat{y}_j \), so \( \hat{y}_j = \hat{y}_j \), which is distributed the same as \( \hat{y}_j \). B2 is uniformly random in \( \mathbb{Z}_p^* \). B is uniformly random in \( \mathbb{Z}_p^* \). C is uniformly random in \( \mathbb{Z}_p^* \). B computes \( \tilde{h}_j = C^{q_j}, \tilde{g}_j = (B^{x_8+i})^{y_1} = \tilde{y}_j^{y_1} \), which is distributed the same as \( \tilde{y}_j^{y_1} \).
The only difference between the two games is that in Game 4, the polynomials \( p_j(x) \) are evaluated at \( \alpha \), which is independent of the true secret point \( \hat{x} \). If \( \hat{q}_i = \hat{q}_j \) for some \( p_i(x) \neq p_j(x) \), then \( R(\hat{q}_i) = R(\hat{q}_j) \), so \( A \) learns that \( p_i(\alpha) = p_j(\alpha) \). The value \( \alpha \) is independent of the adversary's view unless such a collision occurs. We will show that a collision occurs with negligible probability by induction on the number of queries.

For the base case, suppose \( \hat{q}_1 = \hat{q}_2 \). Then, \( \alpha \) is a root of the difference polynomial \( p_1(x) - p_2(x) \). \( A \)'s probability of successfully constructing a difference polynomial with root \( \alpha \) is maximized by choosing \( n \) distinct roots for it. The probability that one of these \( n \) distinct roots is \( \alpha \) is \((n-1)/p\). Thus, the probability that \( \hat{q}_1 = \hat{q}_2 \) is at most \((n-1)/p\), which is negligible. For the induction step, suppose \( \forall i \leq t, \forall j \leq t, \hat{q}_i \neq \hat{q}_j \). The probability that \( \hat{q}_{t+1} \) collides with one of the first \( t \) \( \hat{q} \)'s, conditioned on the fact that there are no collisions among the first \( t \) \( \hat{q} \)'s, is at most \((t+1)(n-1)/p\), which is negligible, completing the induction step.

Thus, \( A \) can distinguish Game 4 from Game 3 only if a collision \( p_i(\hat{x}) = p_j(\hat{x}) \) occurs in Game 3 with non-negligible probability. We now show that such a collision occurs in Game 3 with negligible probability or the DL assumption doesn't hold.

Claim 5. A collision \( p_i(\hat{x}) = p_j(\hat{x}) \) occurs in Game 3 with negligible probability under the DL assumption in \( \mathbb{G}_2 \).

We wish to show that if there exists a PPT adversary \( A \) that produces a collision \( p_i(\hat{x}) = p_j(\hat{x}) \) for some polynomials \( p_i(x) \neq p_j(x) \) with non-negligible probability, then we can construct a PPT adversary \( A' \) that breaks the DL assumption.

Suppose there exists such a PPT algorithm \( A \). Then, we construct a PPT adversary \( A' \) as a reduction \( B \) running \( A \) as a subroutine. We construct the reduction \( B \) for breaking the DL assumption as follows.

\( B \) receives as input \((\hat{A}, \hat{B}) \in \mathbb{G}_2^2 \), where implicitly \( \hat{B} = \hat{A} \hat{x} \) for some uniformly random \( \hat{x} \in \mathbb{Z}_p^* \). (Note that this variant of the DL assumption is equivalent to the one in which \( \hat{x} \) is drawn from \( \mathbb{Z}_p \).)

\( B \) computes the public parameters \( PP \) and keys \((pk, sk) = ((\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5), (x_2, x_3, x_4, x_5)) \) for a mercurial signature scheme \( MS_f \) on messages of length \( \ell = 5 \). \( B \) chooses uniformly at random secret values \( y_1, y_2 \in \mathbb{Z}_p^* \) but does not know the secret point \( \hat{x} \). He also picks \( x_1, x_8 \in \mathbb{Z}_p^* \) and sets \( x_2 = x_8 \cdot y_1 \) and \( x_{10} = x_8 \cdot y_2 \). He then sets \( pk_\alpha = (pk, \hat{A}, \hat{B}, \hat{X}_6, \hat{X}_9, \hat{X}_{10}) \), where \( \hat{X}_i = \hat{P}^{x_i} \), and forwards \( PP_X \) and \( pk_\alpha \) to \( A \).

\( A \) proceeds to make queries to the signing oracle. Acting as the challenger for \( A \), \( B \) is responsible for computing the responses to the signature queries and forwarding them to \( A \). \( B \) responds to the signature queries as follows.

Let \( A \)'s \( j \)th signature query be on message \( m_j = (\tilde{g}_j, u_{j,1}, \ldots, u_{j,n}) \). \( B \) acts as the extractor while \( A \) gives a ZKPoK that, for all \( 1 \leq i \leq n \), he knows \( m_{j,i} \) such that \( u_{j,i} = \tilde{g}_j^{m_{j,i}} \). \( B \) extracts the \( m_{j,i} \)'s, or if the extraction fails, \( B \) denies \( A \) the signature. Otherwise, \( B \) computes the polynomial \( p_j(x) = m_{j,1} + m_{j,2}x + \cdots + m_{j,n}x^{n-1} \).

For all \( 1 \leq t < j \), \( B \) computes the difference polynomial \( p_j(x) - p_t(x) \) and finds its \( n-1 \) roots \( r_{t,1}, \ldots, r_{t,n-1} \). Since \( B \) knows \( \hat{A} \), he can compute \( \hat{A}^{t+1} \) \( \forall t \) and check if \( \hat{A}^{t+1} = \hat{B} \). If this holds for some \( r_{t,j} \), then \( r_{t,j} = \hat{x} \) and \( B \) wins the DL game. If this does not hold, \( B \) picks uniformly at random \( \hat{R}_j \in \mathbb{Z}_p^* \) and computes \( \hat{h}_j = \tilde{g}_j^{\hat{R}_j} \) since he cannot correctly compute \( g^{R(q_j)} \); however, note that \( A \)'s view is identical because he receives random values. \( B \) picks uniformly at random \( w_j \in \mathbb{Z}_p^* \) and computes \( \tilde{g}_j = \hat{g}_j^{w_j}, \hat{h}_j = \hat{h}_j^{w_j} \), and \( u_{j,i} = u_{j,i}^{w_j} \forall i \). He also computes \( \tilde{g}_j^2, \ldots, \tilde{g}_j^n \). He then signs \( n \) messages of the form \( M_{j,1} = (\tilde{g}_j, \tilde{g}_j^{w_j}, \hat{h}_j, u_{j,1}, u_{j,n}) \) using his secret key \( sk \) for \( MS_f \) and sends \( \tilde{m}_j = (\tilde{g}_j, u_{j,1}, \ldots, u_{j,n}) \) and \( (\hat{h}_j, \hat{g}_j, \{\sigma_{j,1}, \ldots, \sigma_{j,n}\}) \) to \( A \), along with a simulated ZKPoK that \( \hat{h}_j \) was computed correctly. \( A \) issues queries for signatures on messages \( \alpha \) a polynomial number of times. \( A \)'s success in producing a difference polynomial \( p_j(x) - p_t(x) \) with root \( \hat{x} \) with non-negligible probability translates into \( B \)'s success in breaking the DL assumption.

From Claim 4 and Claim 5, we can conclude that a PPT adversary \( A \) cannot distinguish Game 3 from Game 4, except with negligible probability.

Game 5. In this game, the glue element is \( \tilde{h} = \hat{g}^{R(m_1, \ldots, m_n)} \), where \( R : (\mathbb{Z}_p^n) \to \mathbb{Z}_p^* \) is a random function.

Game 5 is the same as Game 4, except the challenger \( C \) does not choose a 'fake' secret point \( \alpha \in \mathbb{Z}_p^* \) and does not compute or evaluate the polynomial \( p(x) \). Instead, \( C \) chooses a random function \( R : (\mathbb{Z}_p^n) \to \mathbb{Z}_p^* \) and for each signature computes \( \tilde{h} = \hat{g}^{R(m_1, \ldots, m_n)} \). The rest of the signing protocol is carried out as in Game 4.

Claim 6. An adversary's view in Game 4 is the same as it is in Game 5, except with negligible probability.

Let a (possibly unbounded) adversary \( A \)'s \( j \)th signature query be on message \( m_j = (\tilde{g}_j, u_{j,1}, \ldots, u_{j,n}) \), where \( u_{j,i} = \tilde{g}_j^{m_{j,i}} \) \( \forall i \). In Game 5, the challenger \( C \) extracts
the $m_{j,i}$’s, computes $R(m_{j,1}, \ldots, m_{j,n})$ for some random function $R$, and forms the glue element as $\tilde{h}_j = g^R(m_{j,1}, \ldots, m_{j,n})$, where $\tilde{g}_j = g^{\tilde{w}_j}$ for some uniformly random $w_j \in \mathbb{Z}_p^*$. In Game 4, the challenger $C$ extracts the $m_{j,i}$’s, forms the polynomial $p_j(x) = m_{j,1} + m_{j,2}x + \cdots + m_{j,n}x^{n-1}$, and evaluates $\tilde{q}_j = p_j(\alpha)$ at the “fake” secret point $\alpha \in \mathbb{Z}_p^*$. He then computes $R(\tilde{q}_j)$ for some random function $R$ and forms the glue element as $\tilde{h}_j = \tilde{g}_j^{R(\tilde{q}_j)}$, where $\tilde{g}_j = g^{\tilde{w}_j}$ for some uniformly random $w_j \in \mathbb{Z}_p^*$.

If $q_i = q_j$ for some $p_i(x) \neq p_j(x)$, then $R(\tilde{q}_i) = R(\tilde{q}_j)$, so $A$ learns that $p_i(\alpha) = p_j(\alpha)$. The value $\alpha$ is independent of the adversary’s view unless such a collision occurs. We showed in Claim 4 that a collision $p_i(\alpha) = p_j(\alpha)$ occurs in Game 4 with negligible probability. If there are no such collisions, $A$’s view is identical in both games because he receives random values.

This completes the proof of unforgeability for $\text{MS}_X$ (Theorem 4).

### 4.4 Class-hiding

Message class-hiding states that given two messages $m_1$ and $m_2$, it is hard to tell if $m_2 \in [m_1]_{R_m}$. Public key class-hiding states that given two public keys $pk_{X,1}$ and $pk_{X,2}$ and oracle access to the signing algorithm for both of them, it is hard to tell if $pk_{X,2} \in [pk_{X,1}]_{R_m}$.

**Theorem 5** (Message class-hiding). Let $\text{MS}_f$ be a mercurial signature scheme on message space $(G_1^*)^5$ as in Theorem 1, and let $\text{MS}_X$ be the variable-length mercurial signature scheme on message space $(G_1^*)^{n+1}$ constructed above. Then, message class-hiding of $\text{MS}_X$ holds under the decisional Diffie-Hellman assumption (DDH) in $G_1$. The same holds when $G_1$ and $G_2$ are swapped.

The proof is a straightforward hybrid argument inherited from Fuchsbauer et al. [22].

**Theorem 6** (Public key class-hiding). Let $\text{MS}_f$ be a mercurial signature scheme on message space $(G_1^*)^5$, and let $\text{MS}_X$ be the variable-length mercurial signature scheme on message space $(G_1^*)^{n+1}$ constructed above. Suppose all signatures are issued via the interactive signing protocol described in Section 4.1, where the proof system used in Step 1 is extractable under sequential (or concurrent) composition. Then, public key class-hiding of $\text{MS}_X$ holds sequentially (or concurrently) under the DL assumption in $G_2$, the $\text{ABDDH}^+$ assumption in $G_1$, and the DDH assumption in $G_1$ and $G_2$. The same holds when $G_1$ and $G_2$ are swapped.

For the proof, consider two public keys for $\text{MS}_X$:

$$pk_{X,1} = (p_{k_1}, \hat{p}_{x_1,0}, \hat{p}_{x_1,0}, \hat{p}_{x_1,1}, \hat{p}_{x_1,1}, \hat{p}_{y_1,0}^{(1)}, \hat{p}_{y_1,1}^{(1)})$$

$$pk_{X,2} = (p_{k_2}, \hat{p}_{x_2,0}, \hat{p}_{x_2,0}, \hat{p}_{x_2,1}, \hat{p}_{x_2,1}, \hat{p}_{y_2,0}^{(2)}, \hat{p}_{y_2,1}^{(2)})$$

where $x_\delta (\delta, x_\delta, \hat{x}_\delta, y_1^{(\delta)}, y_2^{(\delta)}) \in \mathbb{Z}_p^*$ for $\delta \in \{1, 2\}$. They are independent if these values are sampled uniformly at random from $\mathbb{Z}_p^*$ and equivalent if $pk_{X,2} = pk_{X,1}^\beta$ for some $\beta \in \mathbb{Z}_p^*$. They are said to be 1/2 independent and 1/2 equivalent if $pk_2 = pk_1^\beta$, but the remaining elements are independent.

We construct a sequence of games beginning with the real signing game in which $pk_{X,1}$, $pk_{X,2}$ are independent (Game 0). In the real signing game, a signature query on a message $m$ under chosen public key $pk_{X,\delta}$ for $\delta \in \{1, 2\}$ results in a glue element computed as $\tilde{h} = \tilde{g}_1^{(\delta)} \cdot q_1$, where $q_1 = p(\tilde{x}_\delta)$ and $y_2^{(\delta)} := y_1^{(\delta)} \cdot y_2^{(\delta)}$. The sequence of games ends with the real signing game in which $pk_{X,1}$, $pk_{X,2}$ are equivalent (Game 13). We show that Game 0 and Game 13 are indistinguishable via a sequence of intermediate games. These games cycle through public keys $pk_{X,1}$, $pk_{X,2}$ that are independent, 1/2 independent and 1/2 equivalent, and equivalent, as well as glue elements that are computed in the various ways specified in the proof of unforgeability. Since the real signing game in which $pk_{X,1}$, $pk_{X,2}$ are independent (Game 0) is indistinguishable from the real signing game in which $pk_{X,1}$, $pk_{X,2}$ are equivalent (Game 13), $\text{MS}_X$ satisfies public key class-hiding. The proof can be found in Appendix C.

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### References


A Prior Construction [15]

The prior construction of mercurial signatures is as follows [15]. The message space consists of vectors of group elements from \( G_1^\ell \), the space of secret keys consists of vectors of elements from \( Z_p^* \), and the space of public keys consists of vectors of group elements from \( G_2^\ell \). Once the prime \( p \), \( G_1 \), \( G_2 \), and a fixed length parameter \( \ell \) are well defined, the equivalence relations are as follows:

\[
\begin{align*}
\mathcal{R}_M & = \{ (M, M') \in (G_1^\ell)^\ell \times (G_1^\ell)^\ell \mid \exists \mu \in Z_p^* \text{ s.t. } M' = M^\mu \} \\
\mathcal{R}_{sk} & = \{ (sk, sk') \in (Z_p^*)^\ell \times (Z_p^*)^\ell \mid \exists \rho \in Z_p^* \text{ s.t. } sk = \rho \cdot sk' \} \\
\mathcal{R}_{pk} & = \{ (pk, pk') \in (G_2^\ell)^\ell \times (G_2^\ell)^\ell \mid \exists \rho \in Z_p^* \text{ s.t. } pk = \rho \cdot pk' \}
\end{align*}
\]

The message space for this mercurial signature scheme consists of vectors of group elements from \( G_1^\ell \), but a mercurial signature scheme with message space \( G_2^\ell \) can be obtained by simply switching \( G_1^\ell \) and \( G_2^\ell \) throughout. The algorithms are as follows:

\[
\begin{align*}
\text{PPGen}(1^k) \rightarrow PP: \text{Compute } BG & \leftarrow \text{BGGen}(1^k). \text{ Output } \\
PP = (G_1, G_2, G_T, P, \hat{P}, e).
\end{align*}
\]

\[
\begin{align*}
\text{KeyGen}(PP, \ell) \rightarrow (pk, sk): \text{For } 1 \leq i \leq \ell, \text{ pick } x_i \leftarrow Z_p^*, \\
\text{ set } sk = (x_1, \ldots, x_\ell), \text{ pk} = (\hat{X}_1, \ldots, \hat{X}_\ell), \text{ where } \hat{X}_i \leftarrow \hat{P}^{x_i} \text{ for } 1 \leq i \leq \ell. \text{ Output } (pk, sk).
\end{align*}
\]

\[
\begin{align*}
\text{Sign}(sk, M) \rightarrow \sigma: \text{On input } sk = (x_1, \ldots, x_\ell), \text{ M} = (M_1, \ldots, M_\ell) \in (G_1^\ell)^\ell, \text{ sample } y \leftarrow Z_p^*, \text{ output } \sigma = (Z, Y, \hat{Y}), \text{ where } Z = \left( \prod_{i=1}^{\ell} M_i^{x_i} \right)^{\psi}, Y = P^\frac{y}{Z}, \text{ and } \\
\hat{Y} = \hat{P}^\frac{y}{Z}.
\end{align*}
\]

\[
\begin{align*}
\text{Verify}(pk, M, \sigma) \rightarrow 0/1: \text{On input } pk = (\hat{X}_1, \ldots, \hat{X}_\ell), \text{ M} = (M_1, \ldots, M_\ell), \text{ and } \sigma = (Z, Y, \hat{Y}), \text{ check } \prod_{i=1}^{\ell} e(M_i, \hat{X}_i) = e(Z, Y) \wedge e(Y, \hat{P}) = e(P, \hat{Y}). \text{ If this holds, output } 1; \text{ otherwise, output } 0.
\end{align*}
\]

\[
\begin{align*}
\text{ConvertSK}(sk, \rho) \rightarrow sk: \text{On input } sk = (x_1, \ldots, x_\ell) \text{ and } \text{ key converter } \rho \in Z_p^*, \text{ output new } sk = \rho \cdot sk.
\end{align*}
\]

\[
\begin{align*}
\text{ConvertPK}(pk, \rho) \rightarrow pk: \text{On input } pk = (\hat{X}_1, \ldots, \hat{X}_\ell) \text{ and } \text{ key converter } \rho \in Z_p^*, \text{ output new pk} = \rho \cdot pk.
\end{align*}
\]

\[
\begin{align*}
\text{ConvertSig}(pk, M, \sigma, \rho) \rightarrow \hat{\sigma}: \text{On input } pk = (\hat{X}_1, \ldots, \hat{X}_\ell), \text{ key converter } \rho \in Z_p^*, \text{ sample } \psi \leftarrow Z_p^*, \text{ output } \hat{\sigma} = (Z^{\rho \psi}, Y^{\frac{1}{\psi}}, \hat{Y}^{\frac{1}{\psi}}).
\end{align*}
\]

\[
\begin{align*}
\text{ChangeRep}(pk, M, \sigma, \mu) \rightarrow (M', \sigma'): \text{On input } pk, M, \sigma = (Z, Y, \hat{Y}), \text{ } \mu \in Z_p^*, \text{ sample } \psi \leftarrow Z_p^*, \text{ Compute } M' = M^\mu, \sigma' = (Z^{\rho \psi}, Y^{\frac{1}{\psi}}, \hat{Y}^{\frac{1}{\psi}}). \text{ Output } (M', \sigma').
\end{align*}
\]

B Zero-Knowledge Proofs

Let us now address which zero-knowledge proof of knowledge (ZKPoK) ought to be used in the signing protocol (Section 4.1). There is a rich literature on ZKPoK protocols for discrete logarithm-based relations that are both practical and provably secure. For our purposes, a ZKPoK protocol needs to be secure under the appropriate notion of composition: our unforgeability game allows the adversary to issue many signing queries, so the challenger must be able to respond to many queries. The best security for our purposes would be UC security [11], but it may come at an efficiency cost. For efficient and UC-secure \( \Sigma \)-protocols [16], Dodis, Shoup, and Walfish [17] offer a solution, but it relies on verifiable encryption [10] or similar, which adds complexity and setup assumptions. In the random oracle model, Fischlin [19] as well as Bernhard, Fischlin, and Warinschi [3] show how to get an extractor that does not need to rewind, thereby allowing composition. If all we want is sequential composition, then we can rely on the fact that proofs of knowledge compose under sequential composition, but that means that in our unforgeability game, the signer can only respond to one signature query at a time.

At the heart of all of these approaches is an efficient \( \Sigma \)-protocol [16] that is then compiled (using the techniques cited above) into a ZKPoK. Depending on which flavor of ZKPoK is needed, the compiler may be very efficient (e.g., if a Fiat-Shamir proof is good enough) or relatively more involved (e.g., if we want UC security with the Fischlin compiler). Below, we give the \( \Sigma \)-protocols that are needed and refer the reader to the cited literature for the details of how to compile them to obtain a ZKPoK.

The signing protocol features two ZKPoKs. In Step 1, the Receiver of the message on a message \( m = (\hat{g}, u_1, \ldots, u_n) \) gives a ZKPoK that, for all \( 1 \leq i \leq n \), he knows \( m_i \) such that \( u_i = \hat{g}^{m_i} \). This can be instantiated using a standard transformation from a \( \Sigma \)-protocol for proving knowledge of a discrete logarithm (Figure 2) [26].

In Step 3 of the signing protocol, the Signer gives a ZKPoK that he has computed the glue element \( \hat{h} \) correctly. If verification of the glue and signature passes, the Receiver outputs the message \( \hat{m} = (\hat{g}, u_1, \ldots, u_n) \) and signature \( (\hat{h}, \sigma) \). The \( \Sigma \)-protocol for this ZKPoK can be viewed as a combination of the following \( \Sigma \)-protocols: (1) a proof of knowledge of a discrete logarithm; (2) a proof of knowledge of the opening of a commitment (Figure 2); (3) a proof of equality of two committed values; and (4) a proof that a committed value is the product of two other committed values (Figure 3) [24]. This last proof can be used to show that a committed value is the square of another committed value and, furthermore, that a committed value is the \( n^{th} \) power of another.
Clemencher and Michels [9].

Random during setup; the other may be chosen arbitrarily.

Note that for the discrete log assumption: the extractor algorithm either outputs the desired values or solves the instance of the discrete logarithm problem defined by the parameters of the system. 

The honest verifier zero-knowledge property holds for these $\Sigma$-protocols information-theoretically. The knowledge extraction holds under the discrete logarithm assumption: the extractor algorithm either outputs the desired values or solves the instance of the discrete logarithm problem defined by the parameters of the system. 

Note that for the $\Sigma$-protocol proving the equality of committed values, the proof of security goes through as long as one of the commitment keys was chosen uniformly at random during setup; the other may be chosen arbitrarily. This was observed by Clemencher and Michels [9].

We can construct a $\Sigma$-protocol for a ZKPoK of the glue element $\hat{h}$ as follows. The Signer (Prover) engages in following protocols with the Receiver (Verifier):

![Fig. 2. $\Sigma$-protocols for proving knowledge of a discrete logarithm (left-hand side) and knowledge of the opening of a commitment (right-hand side).](image)

1. Prove knowledge of the discrete logarithm of $\hat{X}_6 = \hat{P}^\lambda$ and $\hat{X}_7\hat{X}_6^{-1} = \hat{P}^\lambda$. Let $\hat{X} = \hat{P}^\lambda$ and form a commitment $\hat{C}_z = \hat{P}^{\hat{h}}$. Prove knowledge of the discrete logarithm of $\hat{C}_z\hat{X}^{-1} = \hat{H}^{\hat{r}_z}$. Repeat for $\hat{P}^{\hat{r}_1}, \hat{P}^{\hat{r}_2}, \hat{P}^{\hat{r}_3}$ and $\hat{C}_{y_1} = \hat{P}^{\hat{y}_1}\hat{H}^{\hat{r}_1}$, $\hat{C}_{y_2} = \hat{P}^{\hat{y}_2}\hat{H}^{\hat{r}_2}$. 
2. Form the commitments $\hat{C}_z = \hat{g}^{\hat{h}}$, $\hat{C}_{y_1} = \hat{g}^{\hat{y}_1}$, $\hat{C}_{y_2} = \hat{g}^{\hat{y}_2}$. Prove knowledge of the discrete logarithm of $\hat{g} = \hat{g}^u$ and $\hat{C}_z = \hat{g}^{\hat{h}}$, $\hat{C}_{y_1} = \hat{g}^{\hat{y}_1}$, $\hat{C}_{y_2} = \hat{g}^{\hat{y}_2}$. Prove knowledge of the discrete logarithm of $\hat{g} = \hat{g}^u$ and $\hat{C}_z = \hat{g}^{\hat{h}}$, $\hat{C}_{y_1} = \hat{g}^{\hat{y}_1}$, $\hat{C}_{y_2} = \hat{g}^{\hat{y}_2}$. Prove knowledge of the discrete logarithm of $\hat{g} = \hat{g}^u$ and $\hat{C}_z = \hat{g}^{\hat{h}}$, $\hat{C}_{y_1} = \hat{g}^{\hat{y}_1}$, $\hat{C}_{y_2} = \hat{g}^{\hat{y}_2}$.

![Fig. 3. $\Sigma$-protocols for proving the equality of committed values (left-hand side) and that a committed value is the product of two other committed values (right-hand side).](image)

$$C_w \hat{g}^{-1} = H^{r_w}. \text{ Prove the equality of the committed values in } \hat{C}_z = \hat{P}^{\hat{h}} \hat{H}^{\hat{r}_z} \text{ and } \hat{C}_z = \hat{g}^{\hat{h}} \hat{H}^{\hat{r}_z}. $$

3. Form the following commitments:

$$C_1 = u_1 w^{y_1} y_2 H^{r_1}$$
$$C_2 = u_2 w^{y_1} y_2 z H^{r_2}$$
$$:$$
$$C_{n-1} = u_{n-1} w^{y_1} y_2 z^{n-2} H^{r_{n-1}}$$

and prove that they are products of the contents of the commitments $C_w, C_{y_1}, C_{y_2}, C_z$. (Note that the $C_i$’s may be computed using $u_i$’s as bases because their counterparts in the proofs of equality contain bases chosen from the public parameters.) Now compute $C_n = u_w w^{y_1} y_2 z^{n-1} H^{r_n}$, where $r_n = - \sum_{i=1}^{n-1} r_i$. Then:

$$\prod_{i=1}^{n} C_i = \left( \prod_{i=1}^{n} a_i^{\hat{z}_i^{-1}} \right)^y = \hat{h}$$
Lemma 2. Under the discrete logarithm assumption, the protocol described for computing the glue element $h$ is a $\Sigma$-protocol zero-knowledge proof of knowledge.

Proof. Since all of the $\Sigma$-protocols used are proofs of knowledge, the appropriate values can be extracted. As for the zero-knowledge property, form each commitment $C_1, \ldots, C_{n-1}$ at random. Then set

$$C_n = \frac{h}{\prod_{i=1}^{n-1} C_i}.$$  

Next, invoke the zero-knowledge simulator of all of the constituent $\Sigma$-protocols. $\square$

C Public Key Key-Hiding Proof

Proof. We now provide descriptions of the games and proofs of the claims in Section 4.4.

Game 0. In this real signing game, the public keys $pk_{x_1}, pk_{x_2}$ are independent, and the glue element is $h = \tilde{g}^{(\delta)q_\delta}$, where $q_\delta = p(\tilde{x}_\delta)$ for $\delta \in \{1, 2\}$. There is no extraction or zero-knowledge simulation.

The challenger $C$ computes the public parameters $PP = BG = (G_1, G_2, G_T, \beta, P, \delta, e)$ and two sets of keys for a mercurial signature scheme $MS_f$ on messages of length $\ell = 5$: $(sk_{x_1}, pk_{x_1}) = ((x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}), (\tilde{X}_{1,1}, \tilde{X}_{1,2}, \tilde{X}_{1,3}, \tilde{X}_{1,4}, \tilde{X}_{1,5}))$, $(sk_{x_2}, pk_{x_2}) = ((x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, x_{2,5}), (\tilde{X}_{2,1}, \tilde{X}_{2,2}, \tilde{X}_{2,3}, \tilde{X}_{2,4}, \tilde{X}_{2,5}))$, where $\tilde{X}_{i,j} = \tilde{P}_{x_{i,j}}$. $C$ chooses uniformly at random secret points $\tilde{x}_1, \tilde{x}_2 \in Z_p$ and secret seeds $y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)} \in Z_p$. He also picks $x_{1,6}, x_{1,7}, x_{1,8}, x_{1,9} \in Z_p$ and sets:

$$x_{1,7} = x_{1,6} \cdot \tilde{x}_1, x_{1,9} = x_{1,8} \cdot y_1^{(1)}, x_{1,10} = x_{1,8} \cdot y_2^{(1)}$$

$$x_{2,7} = x_{2,6} \cdot \tilde{x}_2, x_{2,9} = x_{2,8} \cdot y_1^{(2)}, x_{2,10} = x_{2,8} \cdot y_2^{(2)}.$$  

$C$ then sets: $pk_{x_1} = (pk_{x_1}, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10})$, $pk_{x_2} = (pk_{x_2}, \tilde{X}_{2,6}, \tilde{X}_{2,7}, \tilde{X}_{2,8}, \tilde{X}_{2,9}, \tilde{X}_{2,10})$, where $\tilde{X}_{i,j} = \tilde{P}_{x_{i,j}}$. $C$ forwards $PP = PP$ and $pk_{x_1}, pk_{x_2}$ to $A$.

$A$ proceeds to make signature queries on messages of the form $m = (\tilde{g}, u_1, \ldots, u_n) \in (G_1)^{n+1}$, where $\tilde{g}$ is a generator of $G_1$. For each signature query, $A$ selects whether he would choose to be signed under $sk_{x_1}$ or $sk_{x_2}$. $C$ acts as the verifier while $A$ gives a ZKPoK that, for all $1 \leq i \leq n$, he knows $m_i$ such that $u_i = \tilde{g}^{m_i}$. If the verification fails, $C$ denies the signature; otherwise, $C$ computes $y_1^{(1)} = y_1^{(1)} \cdot y_2^{(1)}$ and $y_2^{(2)} = y_1^{(2)} \cdot y_2^{(2)}$ and:

$$h = \left(\prod_{i=1}^{n} u_i^{\delta} \right)^{\tilde{y}_\delta^{(\delta)}},$$  

where $\delta \in \{1, 2\}$ corresponds to the secret key $sk_{x_\delta}$. $C$ picks uniformly at random $w \in Z_p$ and computes $\tilde{g} = \tilde{g}^w, h = \tilde{h}^w$, and $\tilde{u}_i = u_i^w \forall i$. He also computes $\tilde{g}^2, \ldots, \tilde{g}^n$. He then signs $n$ messages of the form $M_i = (\tilde{g}, \tilde{g}^i, \tilde{h}, \tilde{u}_i)$ using his secret key $sk_\delta$ for $MS_f$ and sends $\tilde{m} = (\tilde{g}, \tilde{u}_1, \ldots, \tilde{u}_n)$ and $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a ZKPoK that $\tilde{h}$ was computed correctly. $A$ issues queries for signatures on messages a polynomial number of times.

Game 1. In this game, the public keys $pk_{x_1}, pk_{x_2}$ are again independent, and the glue element is again $h = \tilde{g}^{(\delta)q_\delta}$, where $q_\delta = p(\tilde{x}_\delta)$ for $\delta \in \{1, 2\}$; however, now there is simulation.

Game 1 is the same as Game 0, except the challenger $C$ simulates the ZKPoK that $\tilde{h}$ was computed correctly.

Claim 1. A PPT adversary cannot distinguish Game 0 from Game 1, except with negligible probability.

The only difference between the two games is zero-knowledge simulation. In Game 1, the challenger simulates the ZKPoK that the glue $\tilde{h}$ was computed correctly, whereas in Game 0, the challenger gives a real ZKPoK. If an adversary could distinguish the two games, it would break the zero-knowledge property. This is the same as Claim 1 in the proof of unforgeability.

Game 2. In this game, the public keys $pk_{x_1}, pk_{x_2}$ are again independent, and the glue element is again $h = \tilde{g}^{(\delta)q_\delta}$, where $q_\delta = p(\tilde{x}_\delta)$ for $\delta \in \{1, 2\}$; however, now there is extraction and simulation.

Game 2 is the same as Game 1, except for each signature query, the challenger $C$ acts as the extractor while $A$ gives a ZKPoK that, for all $1 \leq i \leq n$, he knows $m_i$ such that $u_i = \tilde{g}^{m_i}$. $C$ extracts the $m_i$’s, or if the extraction fails, $C$ denies the signature. $C$ computes $\tilde{h}$ as in Game 1, signs $n$ messages $M_i = (\tilde{g}, \tilde{g}^i, \tilde{g}^\delta, \tilde{h}, \tilde{u}_i)$ using his secret key $sk_\delta$ for $MS_f$, and sends $\tilde{m} = (\tilde{g}, \tilde{u}_1, \ldots, \tilde{u}_n)$ and $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a simulated ZKPoK that $\tilde{h}$ was computed correctly.

Claim 2. A PPT adversary cannot distinguish Game 2 from Game 2, except with negligible probability.

The glue elements $\tilde{h}$ in both games are identical. The only difference between the two games is extraction. In Game 2, the challenger extracts the $m_i$’s to compute the glue $\tilde{h}$, whereas in Game 1, the challenger computes the correct $\tilde{h}$ directly from the $u_i$’s, without extracting the $m_i$’s. If an adversary could distinguish the two games, it would break the knowledge extractor property. This is the same as Claim 2 in the proof of unforgeability.
Game 3. In this game, the public keys $\text{pk}_{X,1}, \text{pk}_{X,2}$ are now half in the same equivalence class and half independent, but the glue element remains $h = \tilde{g}^{\delta_x} q_i$, where $q_i = p(\tilde{x}_i)$ for $\delta \in \{1, 2\}$.

Game 3 is the same as Game 2, except the challenger $C$ computes $\text{pk}_2$ as $\text{pk}_2^b$ for a uniformly random $\beta \leftarrow \mathbb{Z}_p^*$. $C$ computes $h$ as in Game 2, signs $n$ messages $M_i = (\tilde{g}, \tilde{g}^i, \tilde{g}^\delta, \tilde{h}, \tilde{u}_i)$ using his secret key $\text{sk}_A$ for $\text{MS}_f$, and sends $\tilde{m} = (\tilde{g}, \tilde{u}_1, \ldots, \tilde{u}_n)$ and $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a simulated ZKPoK that $h$ was computed correctly.

Claim 3. If a PPT adversary can distinguish Game 2 from Game 3 with non-negligible probability, then public key class-hiding of $\text{MS}_f$ doesn’t hold.

Suppose a PPT adversary $A$ can distinguish Game 2 from Game 3 for $\text{MS}_X$ on messages of length $n^\ell$. Then, we construct a PPT reduction $B$ for breaking public key-class hiding of $\text{MS}_f$ as follows. $B$ receives as input $PP$ and two fixed public keys $\text{pk}_1, \text{pk}_2^b$ for a mercurial signature scheme $\text{MS}_f$ on messages of length $\ell = 5$. His goal is to determine if $\text{pk}_2^b \notin [\text{pk}_1, \mathbb{Z}_p]$ or not. He constructs public keys $\text{pk}_{X,1}, \text{pk}_{X,2}$ as follows: $\text{pk}_{X,1} = (\text{pk}_1, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10})$, $\text{pk}_{X,2}^b = (\text{pk}_2^b, \tilde{X}_{2,6}, \tilde{X}_{2,7}, \tilde{X}_{2,8}, \tilde{X}_{2,9}, \tilde{X}_{2,10})$, where the $\tilde{X}_{i,j}$’s are computed independently. $B$ then forwards $PP_X = PP$ and $\text{pk}_{X,1}, \text{pk}_{X,2}$ to $A$.

For each signature query, $A$ selects whether he would like the message $m$ to be signed under $\text{sk}_{X,1}$ or $\text{sk}_{X,2}$. $B$ extracts the $m_i$’s, or if the extraction fails, $B$ denies $A$ the signature. $B$ computes $h$ as in Game 2/3, forwards $n^\ell$ messages $M_i = (\tilde{g}, \tilde{g}^i, \tilde{g}^{\delta_i}, \tilde{h}, \tilde{u}_i)$ to the appropriate signing oracle, either $\text{Sign}_f(\text{sk}_A, \cdot)$ or $\text{Sign}_f(\text{sk}_B, \cdot)$, and forwards the signature $\tilde{m} = (\tilde{g}, \tilde{u}_1, \ldots, \tilde{u}_n)$ and $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a simulated ZKPoK that $h$ was computed correctly. It is clear that $\text{pk}_{X,2}^b$ is half in the same equivalence class as $\text{pk}_{X,1}$ and half independent (Game 3) if and only if $\text{pk}_2^b \notin [\text{pk}_1, \mathbb{Z}_p]$, so $A$’s success in distinguishing Game 2 from Game 3 translates directly into $B$’s success in breaking public key-class-hiding of $\text{MS}_f$.

Game 4. In this game, the public keys $\text{pk}_{X,1}, \text{pk}_{X,2}$ are again half in the same equivalence class and half independent, but the glue element is $h = \tilde{g}^{R_\delta(q_i)}$, where $q_i = p(\tilde{x}_i)$ and $R_\delta : (\mathbb{Z}_p^*)^n \rightarrow \mathbb{Z}_p^*$ is a random function for $\delta \in \{1, 2\}$.

The challenger $C$ computes the public keys as: $\text{pk}_{X,1} = (\text{pk}_1, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10})$, $\text{pk}_{X,2} = (\text{pk}_2^b, \tilde{X}_{2,6}, \tilde{X}_{2,7}, \tilde{X}_{2,8}, \tilde{X}_{2,9}, \tilde{X}_{2,10})$, where the $\tilde{X}_{i,j}$’s are computed independently. $C$ chooses two random functions $R_1, R_2 : \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p^*$ and computes $\tilde{h} = \tilde{g}^{R_\delta(q_i)}$ according to the secret key $\text{sk}_{X,\delta} A$ selected. He then signs $n$ messages $M_i = (\tilde{g}, \tilde{g}^i, \tilde{g}^\delta, \tilde{h}, \tilde{u}_i)$ using his secret key $\text{sk}_A$ for $\text{MS}_f$, and sends $\tilde{m} = (\tilde{g}, \tilde{u}_1, \ldots, \tilde{u}_n)$ and $(\tilde{h}, \sigma = \{\sigma_1, \ldots, \sigma_n\})$ to $A$, along with a simulated ZKPoK that $h$ was computed correctly.

Claim 4. A PPT adversary cannot distinguish Game 3 from Game 4 under the ABDDH assumption in $\mathbb{G}_1$.

This is very similar to Claim 3 (ABDDH+) in the proof of unforgeability.

Game 5. In this game, the public keys $\text{pk}_{X,1}, \text{pk}_{X,2}$ are again half in the same equivalence class and half independent, but the glue element is $h = \tilde{g}^{R_\delta(q_i)}$, where $q_i = p(\tilde{x}_i)$, $\delta$ is a ‘fake’ secret point, and $R_\delta : (\mathbb{Z}_p^*)^n \rightarrow \mathbb{Z}_p^*$ is a random function for $\delta \in \{1, 2\}$.

Game 5 is the same as Game 4, except the challenger $C$ computes the glue element as $h = \tilde{g}^{R_\delta(q_i)}$, where $\delta \in \{1, 2\}$ corresponding to the secret key $\text{sk}_{X,\delta} A$ selected.

Claim 5. A PPT adversary $A$ cannot distinguish Game 4 from Game 5 under the DL assumption in $\mathbb{G}_2$.

The only difference between the two games is that in Game 5, the polynomials $p_j(x)$ are evaluated at a ‘fake’ secret point $\tilde{x}_i$, which is independent of the true secret point $\tilde{x}_i$. If $\tilde{x}_i = q_{\delta,i}$ for some $p_j(x) \neq p_j(x)$ and some $\delta \in \{1, 2\}$, then $R_\delta(q_{\delta,i}) = R_\delta(q_{\delta,j})$, so $A$ learns that $p_i(\tilde{x}_i) = p_j(\tilde{x}_i)$. We showed in Claim 4 of the proof of unforgeability that a collision $p_\alpha = p_j(\tilde{x}_i)$ occurs with negligible probability, so $A$ can distinguish Game 4 from Game 5 only if a collision $p_i(\tilde{x}_i) = p_j(\tilde{x}_i)$ occurs in Game 4 with non-negligible probability. We showed in Claim 5 of the proof of unforgeability that such a collision occurs with negligible probability, or the DL assumption doesn’t hold.

Game 6. In this game, the public keys $\text{pk}_{X,1}, \text{pk}_{X,2}$ are again half in the same equivalence class and half independent, but the glue element is $h = \tilde{g}^{R_\delta(m_1, \ldots, m_n)}$, where $R_\delta : (\mathbb{Z}_p^*)^n \rightarrow \mathbb{Z}_p^*$ is a random function for $\delta \in \{1, 2\}$.

Game 6 is the same as Game 5, except the challenger $C$ computes the glue element as $h = \tilde{g}^{R_\delta(m_1, \ldots, m_n)}$, where $\delta \in \{1, 2\}$ corresponds to the secret key $\text{sk}_{X,\delta} A$ selected.

Claim 6. An adversary’s view in Game 5 is the same as it is in Game 6, except with negligible probability.
If $\tilde{q}_{\delta,i} = \tilde{q}_{\delta,j}$ for some $p_i(x) \neq p_j(x)$ and some $\delta \in \{1, 2\}$, then $R_\delta(\tilde{q}_{\delta,i}) = R_\delta(\tilde{q}_{\delta,j})$, so $A$ learns that $p(\alpha_3) = p_j(\alpha_3)$. The value $\alpha_3$ is independent of the adversary’s view unless such a collision occurs. We showed in Claim 4 of the proof of unforgeability that a collision $p(\alpha_3) = p_j(\alpha_3)$ occurs with negligible probability. If there are no such collisions, $A$’s view is identical in both games because he receives random values.

**Game 7.** In this game, the public keys $\mathbf{pk}_{X,1}$, $\mathbf{pk}_{X,2}$ are again half in the same equivalence class and half independent, but the glue element is $\tilde{h} = \tilde{g}^{R(m_1,\ldots,m_n)}$, where $R : (Z_p^n) \rightarrow Z_p$ is a random function.

Game 7 is the same as Game 6, except the challenger $C$ computes the glue element as $\tilde{h} = \tilde{g}^{R(m_1,\ldots,m_n)}$. Note that the same function $R$ is used regardless of which secret key $\mathbf{sk}_{X,\delta} A$ selected.

**Claim 7.** A PPT adversary cannot distinguish Game 6 from Game 7 under the DDH assumption in $\mathbb{G}_1$.

We prove this via a hybrid argument. Suppose a PPT adversary $A$ can distinguish hybrids $H_i$ from $H_{i+1}$ (described below) for some $i$ with non-negligible probability (bounded by the best advantage in breaking DDH). Then, we construct a PPT reduction $B$ for breaking the DDH assumption as follows. $B$ receives as input $(g_0, A, B, C)$, where $g_0$ is a generator of $\mathbb{G}_1$ and implicitly $A = g_0^a, B = g_0^b$, and $C = g_0^b$ or $g_0^b$ for some uniformly random $a, b, r \in Z_p^n$. $B$ computes public parameters $PP$ and public keys $\mathbf{pk}_{X,1}, \mathbf{pk}_{X,2}$ as follows: $\mathbf{pk}_{X,1} = (\mathbf{pk}_1, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10}), \mathbf{pk}_{X,2} = (\mathbf{pk}_1^\beta, \tilde{X}_{2,6}, \tilde{X}_{2,7}, \tilde{X}_{2,8}, \tilde{X}_{2,9}, \tilde{X}_{2,10})$, where the $\tilde{X}_{i,j}$’s are computed independently. $B$ chooses random functions $R_1, R_2 : (Z_p^n) \rightarrow Z_p^n$ and forwards $PP_X = PP$ and $\mathbf{pk}_{X,1}, \mathbf{pk}_{X,2}$ to $A$.

Let $A$’s $j^{th}$ signature query be on message $m_j = (\tilde{g}_j, u_{j,1},\ldots,u_{j,n})$. $B$ acts as the extractor while $A$ gives a ZKPoK that, for all $1 \leq i \leq n$, he knows $m_{j,i}$ such that $u_{j,i} = \tilde{g}^{m_{j,i}}$. $B$ extracts the $m_{j,i}$’s, or if the extraction fails, $B$ modifies the signature otherwise.

**Claim 8.** A PPT adversary cannot distinguish Game 7 from Game 8 under the DDH assumption in $\mathbb{G}_2$.

Consider the following set of games. In each game, $H_i$ corresponds to the game in which all glue elements are formed as $\tilde{h}_j = \tilde{g}_j^{R(m_1,\ldots,m_n)}$ (Game 7), while $H_{i+1}$ corresponds to the game in which all glue elements are formed as $\tilde{h}_j = \tilde{g}_j^{R(m_1,\ldots,m_n)}$ for $\delta \in \{1, 2\}$ (Game 6). $C = g_0^b$ corresponds to hybrid $H_i$ and $C = g_0^b$ corresponds to hybrid $H_{i+1}$. Thus, if $A$ is able to distinguish $H_i$ from $H_{i+1}$ for some $i$ with non-negligible probability, then $B$ breaks the DDH assumption.

**Game 8.** In this game, now $\mathbf{pk}_{X,2} \in [\mathbf{pk}_{X,1}]_{R_0}$, but the glue element remains $\tilde{h} = \tilde{g}^{R(m_1,\ldots,m_n)}$, where $R : (Z_p^n) \rightarrow Z_p$ is a random function.

Game 8 is the same as Game 7, except the challenger $C$ computes the public keys as $\mathbf{pk}_{X,2} = \mathbf{pk}_{X,1}^\beta$ for a uniformly random $\beta \leftarrow Z_p^n$.

**Claim 9.** A PPT adversary cannot distinguish Game 7 from Game 8 under the DDH assumption in $\mathbb{G}_2$.

Recall that $\mathbf{pk}_{X,1}$ and $\mathbf{pk}_{X,2}$ are of the form:

$\mathbf{pk}_{X,1} = (\mathbf{pk}_1, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8, \tilde{X}_{1,9}}, \tilde{X}_{1,10}), \mathbf{pk}_{X,2} = (\mathbf{pk}_1^\beta, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10}),$ where $\beta, \gamma, \lambda, \tilde{x}_2, y_1^{(2)}(1), y_2^{(2)}(1), y_2^{(2)}(2) \leftarrow Z_p^n$ are all uniformly random.

**Intermediate Game 1.** Consider $\mathbf{pk}_{X,1}$ and $\mathbf{pk}_{X,2}$ of the form: $\mathbf{pk}_{X,1} = (\mathbf{pk}_1, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10}), \mathbf{pk}_{X,2} = (\mathbf{pk}_1, \tilde{X}_{1,6}, \tilde{X}_{1,7}, \tilde{X}_{1,8}, \tilde{X}_{1,9}, \tilde{X}_{1,10}),$ where $\beta, \gamma, \lambda, y_1^{(1)}, y_1^{(2)}(1), y_2^{(2)}(1), y_2^{(2)}(2) \leftarrow Z_p^n$ are all uniformly random.
The reduction plugs in the DH challenge \((\hat{g}_0, \hat{A}, \hat{B}, \hat{C})\) as follows: \(pk_{X,1} = (pk_{k,1}, \hat{g}_0, \hat{A}, \hat{X}_1, \hat{X}_2, \hat{Y}_1, \hat{Y}_2)\)

\(pk_{X,2} = (pk_{k,2}, \hat{B}, \hat{C}, \hat{X}_3, \hat{X}_4, \hat{Y}_3, \hat{Y}_4)\)

\(\beta, \gamma, y_1, y_2, z_1, z_2 \leftrightarrow Z_p^*\) are all uniformly random.

Thus, we have that \(\hat{x}_1 = a\) and \(\gamma = b\). If \(\hat{C} = \hat{g}_0^b\), then \(\hat{x}_2 = a = \hat{x}_1\) (Int. Game 1). If \(\hat{C} = \hat{g}_0^a\), then \(\hat{x}_1\) and \(\hat{x}_2\) are independent (Game 7).

**Intermediate Game 2.** Consider \(pk_{X,1}\) and \(pk_{X,2}\) of the form: \(pk_{X,1} = (pk_{k,1}, \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_6, \hat{X}_7, \hat{X}_8, \hat{X}_9, \hat{X}_{10}, \hat{X}_{11})\)

where \(\beta, \gamma, y_1, y_2 \leftrightarrow Z_p^*\) are all uniformly random.

The reduction plugs in the DH challenge \((g_0, A, B, C)\) as follows: \(pk_{X,1} = (pk_{k,1}, \hat{X}_1, \hat{X}_2, g_0, A, g_0^a)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_6, \hat{X}_7, B, B, C)\)

Thus, we have that \(y_1 = a\) and \(\lambda = b\). If \(\hat{C} = \hat{g}_0^b\), then \(\hat{y}_2 = \hat{y}_1\) (Game 2). If \(\hat{C} = \hat{g}_0^a\), then \(\hat{y}_1\) and \(\hat{y}_2\) are independent (Int. Game 1).

**Intermediate Game 3.** Consider \(pk_{X,1}\) and \(pk_{X,2}\) of the form: \(pk_{X,1} = (pk_{k,1}, \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5, \hat{X}_6, \hat{X}_7)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_8, \hat{X}_9, \hat{X}_{10}, \hat{X}_{11}, \hat{X}_{12}, \hat{X}_{13})\)

where \(\beta, \gamma, \lambda \leftrightarrow Z_p^*\) are all uniformly random.

The reduction plugs in the DH challenge \((g_0, A, B, C)\) as follows: \(pk_{X,1} = (pk_{k,1}, \hat{X}_1, \hat{X}_2, g_0, g_0^a, A)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_3, \hat{X}_4, B, B, C)\)

Thus, we have that \(y_2 = a\) and \(\lambda = b\). If \(\hat{C} = \hat{g}_0^b\), then \(\hat{y}_2 = \hat{y}_1\) (Game 3). If \(\hat{C} = \hat{g}_0^a\), then \(\hat{y}_1\) and \(\hat{y}_2\) are independent (Int. Game 2).

**Intermediate Game 4.** Consider \(pk_{X,1}\) and \(pk_{X,2}\) of the form: \(pk_{X,1} = (pk_{k,1}, \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5, \hat{X}_6, \hat{X}_7)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_8, \hat{X}_9, \hat{X}_{10}, \hat{X}_{11}, \hat{X}_{12}, \hat{X}_{13})\)

where \(\beta, \gamma \leftrightarrow Z_p^*\) are all uniformly random.

The reduction plugs in the DH challenge \((g_0, A, B, C)\) as follows: \(pk_{X,1} = (pk_{k,1}, \hat{g}_0, \hat{g}_0^a, B, B, C)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_3, \hat{X}_4, B, B, C)\)

Thus, we have that \(\gamma = a\). If \(\hat{C} = \hat{g}_0^b\), then \(\lambda = a = \gamma\) (Int. Game 4). If \(\hat{C} = \hat{g}_0^a\), then \(\hat{C}\) is distributed the same as \(B^a\) for \(\lambda\) independent from \(\gamma\) (Int. Game 3).

**Game 8.** Recall that \(pk_{X,1}\) and \(pk_{X,2}\) are of the form: \(pk_{X,1} = (pk_{k,1}, \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5, \hat{X}_6, \hat{X}_7)\)

\(pk_{X,2} = (pk_{k,2}, \hat{X}_8, \hat{X}_9, \hat{X}_{10}, \hat{X}_{11}, \hat{X}_{12}, \hat{X}_{13})\)

where \(\beta \leftrightarrow Z_p^*\) is uniformly random.

The reduction plugs in the DH challenge \((\hat{g}_0, A, B, C)\) as follows: \(pk_{X,1} = \frac{(\hat{g}_0, \hat{g}_0^a, \hat{g}_0^b, \hat{g}_0, \hat{g}_0^a, \hat{g}_0, \hat{g}_0^a, \hat{g}_0^b)}{\hat{g}_0^a}\)

\((\hat{g}_0, \hat{g}_0^a, \hat{g}_0^b, \hat{g}_0, \hat{g}_0^a, \hat{g}_0, \hat{g}_0^a, \hat{g}_0^b)\) were independent from \(b\) (Int. Game 4).

**Game 8.** \(pk_{X,2} \in [pk_{X,1}]_{\pi_\mu}, \hat{h} = \hat{g}^{R\{m_1, ..., m_n\}}\).

**Claim 9:** polynomial collision argument, same as unforgeability Claim 6

**Game 9.** \(pk_{X,2} \in [pk_{X,1}]_{\pi_\mu}, \hat{h} = \hat{g}^{R\{q\}}\).

**Claim 10:** polynomial collision argument and DL assumption in \(G_2\), similar to unforgeability Claims 4/5

**Game 10.** \(pk_{X,2} \in [pk_{X,1}]_{\pi_\mu}, \hat{h} = \hat{g}^{\mu\cdot q}\).

**Claim 11:** ABDDH, same as unforgeability Claim 3

**Game 11.** \(pk_{X,2} \in [pk_{X,1}]_{\pi_\mu}, \hat{h} = \hat{g}^{\mu\cdot q}\).

**Claim 12:** knowledge extractor property, same as unforgeability Claim 2

**Game 12.** \(pk_{X,2} \in [pk_{X,1}]_{\pi_\mu}, \hat{h} = \hat{g}^{\mu\cdot q}\). No extraction.

**Claim 13:** ZK property, same as unforgeable Claim 1

**Game 13.** \(pk_{X,2} \in [pk_{X,1}]_{\pi_\mu}, \hat{h} = \hat{g}^{\mu\cdot q}\). No extraction or ZK simulation. This is the real signing game.

\[\square\]